

Local boundedness, maximum principles, and continuity of solutions to infinitely degenerate elliptic equations

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ABSTRACT. We obtain local boundedness and maximum principles for weak subsolutions to certain infinitely degenerate elliptic divergence form equations, and also continuity of weak solutions in some cases. For example, we consider the family $\{f_{k,\sigma}\}_{k \in \mathbb{N}, \sigma > 0}$ with

$$f_{k,\sigma}(x) = |x|^{\left(\ln^{(k)} \frac{1}{|x|}\right)^\sigma}, \quad -\infty < x < \infty,$$

of infinitely degenerate functions at the origin, and derive conditions on the parameters k and σ under which all weak solutions to the associated infinitely degenerate quasilinear equations of the form

$$\operatorname{div} A(x, y, u) \operatorname{grad} u = \phi(x, y), \quad A(x, y, z) \sim \begin{bmatrix} 1 & 0 \\ 0 & f_{k,\sigma}(x)^2 \end{bmatrix},$$

with rough data A and ϕ , are locally bounded / satisfy a maximum principle / are continuous.

As an application we obtain weak hypoellipticity (i.e. smoothness of all weak solutions) of certain *infinitely* degenerate quasilinear equations

$$\frac{\partial u}{\partial x^2} + f(x, u(x, y))^2 \frac{\partial u}{\partial y^2} = \phi(x, y),$$

with smooth data $f(x, z) \sim f_{k,\sigma}(x)$ and $\phi(x, y)$ where $f(x, z)$ has a sufficiently mild nonlinearity and degeneracy.

We also consider extensions of these results to \mathbb{R}^3 and obtain some limited sharpness. In order to prove these theorems we develop subrepresentation inequalities for these geometries and obtain corresponding Poincaré and Orlicz-Sobolev inequalities. We then apply more abstract results (that hold also in higher dimensional Euclidean space) in which these Poincaré and Orlicz-Sobolev inequalities are assumed to hold.

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Preface

There is a large and well-developed theory of elliptic and subelliptic equations with rough data, and also a smaller theory still in its infancy of infinitely degenerate elliptic equations with smooth data. Our purpose here is to initiate a study of the DeGiorgi-Moser regularity theory in the context of equations that are both infinitely degenerate elliptic and have rough data. This monograph can be viewed as taking the first baby steps in what promises to be an exciting investigation in view of the numerous surprises encountered here in the implementation of DeGiorgi-Moser iteration in the infinitely degenerate regime. The parallel approach of Nash seems difficult to adapt to the infinitely degenerate case, but remains an enticing possibility for future research.

Part 1

Overview

The regularity theory of subelliptic linear equations with smooth coefficients is well established, as evidenced by the results of Hörmander [Ho] and Fefferman and Phong [FePh]. In [Ho], Hörmander obtained hypoellipticity of sums of squares of smooth vector fields whose Lie algebra spans at every point. In [FePh], Fefferman and Phong considered general nonnegative semidefinite smooth linear operators, and characterized subellipticity in terms of a containment condition involving Euclidean balls and "subunit" balls related to the geometry of the nonnegative semidefinite form associated to the operator.

The theory in the infinite regime however, has only had its surface scratched so far, as evidenced by the results of Fedii [Fe] and Kusuoka and Strook [KuStr]. In [Fe], Fedii proved that the two-dimensional operator $\frac{\partial}{\partial x^2} + f(x)^2 \frac{\partial}{\partial y^2}$ is hypoelliptic merely under the assumption that f is smooth and positive away from $x = 0$. In [KuStr], Kusuoka and Strook showed that under the same conditions on $f(x)$, the three-dimensional analogue $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + f(x)^2 \frac{\partial^2}{\partial z^2}$ of Fedii's operator is hypoelliptic *if and only if* $\lim_{x \rightarrow 0} x \ln f(x) = 0$. These results, together with some further refinements of Christ [Chr], illustrate the complexities associated with regularity in the infinite regime, and point to the fact that the theory here is still in its infancy.

The problem of extending these results to include quasilinear operators requires an understanding of the corresponding theory for linear operators with *nonsmooth* coefficients, generally as rough as the weak solution itself. In the elliptic case this theory is well-developed and appears for example in Gilbarg and Trudinger [GiTr] and many other sources. The key breakthrough here was the Hölder *a priori* estimate of DeGiorgi, and its later generalizations independently by Nash and Moser. The extension of the DeGiorgi-Nash-Moser theory to the subelliptic or finite type setting, was initiated by Franchi [Fr], and then continued by many authors, including one of the present authors with Wheeden [SaWh4].

The subject of the present monograph is the extension of DeGiorgi-Moser theory to the infinitely degenerate regime. Our theorems fall into two broad categories. First, there is the *abstract theory* in all dimensions, in which we assume appropriate Orlicz-Sobolev inequalities and deduce local boundedness and maximum principles for weak subsolutions, and also continuity for weak solutions. This theory is complicated by the fact that the companion Cacciopoli inequalities are now far more difficult to establish for iterates of the Young functions that arise in the Orlicz-Sobolev inequalities. Second, there is the *geometric theory* in dimensions two and three, in which we establish the required Orlicz-Sobolev inequalities for large families of infinitely degenerate geometries, thereby demonstrating that our abstract theory is not vacuous, and that it does in fact produce new theorems.

The results obtained here are of course also in their infancy, leaving many intriguing questions unanswered. For example, our implementation of Moser iteration requires a sufficiently large Orlicz bump, which in turn restricts the conclusions of the method to fall well short of existing counterexamples. It is a major unanswered question as to whether or not this 'Moser gap' is an artificial obstruction to local boundedness. Finally, the contributions of Nash to the classical DeGiorgi-Nash-Moser theory revolve around moment estimates for solutions, and we have been unable to extend these to the infinitely degenerate regime, leaving a tantalizing loose end. We now turn to a more detailed description of these results and questions in the introduction that follows.

CHAPTER 1

Introduction

In 1971 Fedii proved in [Fe] that the linear second order partial differential operator

$$\mathcal{L}u(x, y) \equiv \left\{ \frac{\partial}{\partial x^2} + f(x)^2 \frac{\partial}{\partial y^2} \right\} u(x, y)$$

is *hypoelliptic*, i.e. every distribution solution $u \in \mathcal{D}'(\mathbb{R}^2)$ to the equation $\mathcal{L}u = \phi \in C^\infty(\mathbb{R}^2)$ in \mathbb{R}^2 is smooth, i.e. $u \in C^\infty(\mathbb{R}^2)$, provided:

- $f \in C^\infty(\mathbb{R})$,
- $f(0) = 0$ and f is positive on $(-\infty, 0) \cup (0, \infty)$.

The main feature of this remarkable theorem is that the order of vanishing of f at the origin is unrestricted, in particular it can vanish to infinite order. If we consider the analogous (special form) quasilinear operator,

$$\mathcal{L}_{\text{quasi}}u(x, y) \equiv \left\{ \frac{\partial}{\partial x^2} + f(x, u(x, y))^2 \frac{\partial}{\partial y^2} \right\} u(x, y),$$

then of course $f(x, u(x, y))$ makes no sense for u a distribution, but in the special case where $f(x, z) \approx f(x, 0)$, the appropriate notion of hypoellipticity for $\mathcal{L}_{\text{quasi}}$ becomes that of $W_A^{1,2}(\mathbb{R}^2)$ -hypoellipticity with $A \equiv \begin{bmatrix} 1 & 0 \\ 0 & f(x, 0)^2 \end{bmatrix}$, where we say $\mathcal{L}_{\text{quasi}}$ is $W_A^{1,2}(\mathbb{R}^2)$ -hypoelliptic if every $W_A^{1,2}(\mathbb{R}^2)$ -weak solution u to the equation $\mathcal{L}_{\text{quasi}}u = \phi$ is smooth for all smooth data $\phi(x, y)$. Here $u \in W_A^{1,2}(\mathbb{R}^2)$ is a $W_A^{1,2}(\mathbb{R}^2)$ -weak solution to $\mathcal{L}_{\text{quasi}}u = \phi$ if

$$-\int (\nabla w)^{\text{tr}} \begin{bmatrix} 1 & 0 \\ 0 & f(x, u(x, y))^2 \end{bmatrix} \nabla u = \int \phi w, \quad \text{for all } w \in W_A^{1,2}(\mathbb{R}^2)_0.$$

See below for a precise definition of the degenerate Sobolev space $W_A^{1,2}(\mathbb{R}^2)$, that informally consists of all $w \in L^2(\mathbb{R}^2)$ for which $\int (\nabla w)^{\text{tr}} A \nabla w < \infty$.

There is apparently no known $W_A^{1,2}(\mathbb{R}^2)$ -hypoelliptic quasilinear operator $\mathcal{L}_{\text{quasi}}$ with coefficient $f(x, z)$ that vanishes to *infinite* order when $x = 0$, despite the abundance of results when f vanishes to *finite* order. However, in the infinite vanishing case, if we assume the stronger condition (1) below and *in addition* condition (2) below,

- (1) $\sup_{(x,z) \in (\mathbb{R} \setminus \{0\}) \times \mathbb{R}} \frac{|f(x,z) - f(x,0)|}{f(x,0)} \leq \frac{1}{2}$ and $\lim_{z \rightarrow 0} \sup_{x \in \mathbb{R} \setminus \{0\}} \frac{|\frac{\partial f}{\partial y}(x,z)|}{f(x,0)} = 0$, and
- (2) a $W_A^{1,2}(\mathbb{R}^2)$ -weak solution u to $\mathcal{L}_{\text{quasi}}u = 0$ is *continuous*,

then in 2009 it was shown by Rios, Sawyer and Wheeden in [RSaW2] that $u \in C^\infty(\mathbb{R}^2)$. As a consequence of this and Theorem 26 below on continuity of weak solutions, we obtain that certain

of these quasilinear operators $\mathcal{L}_{\text{quasi}}$ are $W_A^{1,2}(\mathbb{R}^2)$ -hypoelliptic. For $k \geq 1$ and $\sigma > 0$ let

$$(1.1) \quad f_{k,\sigma}(x) \equiv |x|^{(\ln^{(k)} \frac{1}{|x|})^\sigma}, \quad |x| > 0 \text{ sufficiently small.}$$

THEOREM 1. *Suppose that $f(x, z)$ is smooth in \mathbb{R}^2 and that in addition, $f(x, z)$ satisfies (1) and that for either $k \geq 4$ and $\sigma > 0$, or $k = 3$ and $0 < \sigma < 1$, the function $f(x, 0)$ satisfies*

$$(1.2) \quad cf_{k,\sigma}(x) \leq f(x, 0) \leq Cf_{k,\sigma}(x), \quad \text{for small } |x| > 0.$$

Then the quasilinear operator $\mathcal{L}_{\text{quasi}}$ is $W_A^{1,2}(\mathbb{R}^2)$ -hypoelliptic.

REMARK 2. *The local sup norm bounds $\|D^\alpha u\|_{L^\infty}$ on the derivatives of u in Theorem 1 depend only on the constants C, σ in condition (1.2), on the size $\|D^\alpha \phi\|_{L^\infty}$ of the derivatives of ϕ , and on the norm $\|u\|_{W_A^{1,2}(\mathbb{R}^2)}$ of the weak solution u in $W_A^{1,2}(\mathbb{R}^2)$.*

Of course, to prove Theorem 1, it suffices to show that a weak solution u to an equation $\mathcal{L}_{\text{quasi}}u = \phi$ is continuous, since then the result in [RSaW2] gives smoothness - see Section 1 below for details. In the appendix we give an example involving the Monge-Ampère equation in two dimensions to illustrate the limitation of Theorem 1 to quasi-linear equations.

Our method for proving continuity of weak solutions u to $\mathcal{L}_{\text{quasi}}u = \phi$ is to view u as a weak solution to the linear equation

$$\mathcal{L}u(x, y) \equiv \left\{ \frac{\partial}{\partial x^2} + g(x, y)^2 \frac{\partial}{\partial y^2} \right\} u(x, y) = \phi(x, y),$$

where $g(x, y) = f(x, u(x, y))$ and $\phi(x, y)$ need no longer be smooth, but $g(x, y)$ satisfies the estimate

$$\frac{1}{C}f(x, 0) \leq g(x, y) \leq Cf(x, 0), \quad x \in \mathbb{R},$$

and $\phi(x, y)$ is measurable and admissible - see below for definitions. The method we employ is an adaptation of Moser and Bombieri iteration, which splits neatly into local boundedness of weak subsolutions and continuity of weak solutions. The infinite degeneracy of \mathcal{L} forces our adaptation of Moser and Bombieri iteration to use Young functions that fail to be multiplicative, and this results in numerous complications to be overcome, which we briefly discuss below in the remainder of this overview of the paper. But first we mention as further motivation for this approach, that Kusuoka and Strook [KuStr] considered in 1985 the following three dimensional analogue of Fedii's equation,

$$\mathcal{L}_1 \equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + f(x_1)^2 \frac{\partial^2}{\partial x_3^2},$$

and showed the surprising result that when $f(x_1)$ is smooth and positive away from the origin, the smooth linear operator \mathcal{L}_1 is hypoelliptic *if and only if*

$$\lim_{r \rightarrow 0} r \ln f(r) = 0.$$

Thus we will begin with an abstract approach in higher dimensions, where we assume certain Orlicz Sobolev inequalities hold, and then specialize to two and three dimensions where we establish geometries that are sufficient to prove the required Orlicz Sobolev inequalities.

We consider the second order special quasilinear equation (where only u , and not ∇u , appears nonlinearly),

$$(1.3) \quad Lu \equiv \nabla^{\text{tr}} \mathcal{A}(x, u(x)) \nabla u = \phi, \quad x \in \Omega$$

where Ω is a bounded domain in \mathbb{R}^n , and we assume the following structural condition on the quasilinear matrix $A(x, u(x))$,

$$(1.4) \quad k \xi^T A(x) \xi \leq \xi^T \mathcal{A}(x, z) \xi \leq K \xi^T A(x) \xi ,$$

for a.e. $x \in \Omega$ and all $z \in \mathbb{R}$, $\xi \in \mathbb{R}^n$. Here k, K are positive constants and $A(x) = B(x)^{\text{tr}} B(x)$ where $B(x)$ is a Lipschitz continuous $n \times n$ real-valued matrix defined for $x \in \Omega$. We define the A -gradient by

$$\nabla_A = B(x) \nabla ,$$

and the associated degenerate Sobolev space $W_A^{1,2}(\Omega)$ to have norm

$$\|v\|_{W_A^{1,2}} \equiv \sqrt{\int_{\Omega} (|v|^2 + \nabla v^{\text{tr}} A \nabla v)} = \sqrt{\int_{\Omega} (|v|^2 + |\nabla_A v|^2)}.$$

DEFINITION 3. Let Ω be a bounded domain in \mathbb{R}^n . Assume that $\phi \in L_{\text{loc}}^2(\Omega)$. We say that $u \in W_A^{1,2}(\Omega)$ is a weak solution to $Lu = \phi$ provided

$$-\int_{\Omega} \nabla w(x)^{\text{tr}} \mathcal{A}(x, u(x)) \nabla u = \int_{\Omega} \phi w$$

for all $w \in \left(W_A^{1,2}\right)_0(\Omega)$, where $\left(W_A^{1,2}\right)_0(\Omega)$ denotes the closure in $W_A^{1,2}(\Omega)$ of the subspace of Lipschitz continuous functions with compact support in Ω .

Note that our structural condition (1.4) implies that the integral on the left above is absolutely convergent, and our assumption that $\phi \in L_{\text{loc}}^2(\Omega)$ implies that the integral on the right above is absolutely convergent.

Weak sub and super solutions are defined by replacing $=$ with \geq and \leq respectively in the display above. In particular note that if u is a weak sub respectively super solution to $Lu = \phi$, then so is $u^+ \equiv \max\{u, 0\}$ respectively $u^- \equiv \min\{u, 0\}$.

We will consider separately

- local boundedness and maximum principle for weak subsolutions, and
- continuity of weak solutions.

More precisely, we will first obtain *abstract* local boundedness results and maximum principles in which we *assume* appropriate Poincaré and Orlicz-Sobolev inequalities hold. Then we will apply our study of degenerate geometries to prove that these Poincaré and Orlicz-Sobolev inequalities hold in specific situations, thereby obtaining our *geometric* local boundedness results and maximum principles in which we only assume information on the *size* of the degenerate geometries. The techniques used for local boundedness of weak subsolutions and maximum principles are very similar and so are considered together at one time. On the other hand, the techniques required for obtaining *continuity* of weak solutions are more complicated, and thus we consider abstract and geometric theorems for continuity later on.

1. Moser iteration, local boundedness and maximum principle for subsolutions

Let Ω be a bounded domain in \mathbb{R}^n . There is a quadruple $(\mathcal{A}, d, \varphi, \Phi)$ of objects of interest in our abstract local boundedness theorem in Ω , namely

- (1) the matrix $\mathcal{A} = \mathcal{A}(x, z)$ associated with our equation and the A -gradient,
- (2) a metric d giving rise to the balls $B(x, r)$ that appear in our Sobolev inequality, and also in our sequence $\{\psi_j\}_{j=1}^{\infty}$ of accumulating Lipschitz functions,

- (3) a positive function $\varphi(r)$ for $r \in (0, R)$ that appears in place of the radius r in our Sobolev inequality, and
- (4) a Young function Φ appearing in our Sobolev inequality.

We will assume two connections between these objects, namely

- the existence of an appropriate sequence $\{\psi_j\}_{j=1}^\infty$ of accumulating Lipschitz functions that connects two of the objects of interest \mathcal{A} and d , and
- a Sobolev Orlicz bump inequality,

$$\int_B \Phi(w) \leq \Phi(C\varphi(r(B)) \|\nabla_A w\|_{L^1}), \quad \text{supp } w \subset B,$$

that connects all four objects of interest \mathcal{A} , d , φ and Φ .

REMARK 4. *To see what the Sobolev Orlicz bump inequality looks like in a special case, suppose the metric d arises from the metric tensor*

$$dt^2 = dx^2 + \frac{1}{f(x)^2} dy^2$$

for a function $f(x) = e^{-F(x)} > 0$ on $(0, R)$ satisfying the structure conditions in Definition 14 below, and suppose that the Young function $\Phi = \Phi_m$ is given by (1.6) below. Then we will take

$$\varphi(r) \equiv \frac{1}{|F'(r)|} e^{C_m \left(\frac{|F'(r)|^2}{F''(r)} \right)^{m-1}},$$

and refer to the function $\varphi(r) = \varphi_{F,m}(r)$ as the superradius associated with this metric d and Φ_m . We will show below that the Sobolev Orlicz bump inequality holds in this setting provided the superradius $\varphi_{F,m}(r)$ is nondecreasing for $r > 0$ small.

We now describe these matters in more detail.

DEFINITION 5 (Standard sequence of accumulating Lipschitz functions). *Let Ω be a bounded domain in \mathbb{R}^n . Fix $r > 0$ and $x \in \Omega$. We define an (\mathcal{A}, d) -standard sequence of Lipschitz cutoff functions $\{\psi_j\}_{j=1}^\infty$ at (x, r) , along with sets $B(x, r_j) \supset \text{supp } \psi_j$, to be a sequence satisfying $\psi_j = 1$ on $B(x, r_{j+1})$, $r_1 = r$, $r_\infty \equiv \lim_{j \rightarrow \infty} r_j = \frac{1}{2}$, $r_j - r_{j+1} = \frac{c}{j^2} r$ for a uniquely determined constant c , and $\|\nabla_A \psi_j\|_\infty \lesssim \frac{j^2}{r}$ with ∇_A as in (1.10) (see e.g. [SaWh4]).*

We will need to assume the following single scale (Φ, φ) -Sobolev Orlicz bump inequality:

DEFINITION 6. *Let Ω be a bounded domain in \mathbb{R}^n . Fix $x \in \Omega$ and $\rho > 0$. Then the single scale (Φ, φ) -Sobolev Orlicz bump inequality at (x, ρ) is:*

$$(1.5) \quad \Phi^{(-1)} \left(\int_{B(x, \rho)} \Phi(w) d\mu_{x, \rho} \right) \leq C\varphi(\rho) \|\nabla_A w\|_{L^1(\mu_{x, \rho})}, \quad w \in \text{Lip}_0(B(x, \rho)),$$

where $d\mu_{x, \rho}(y) = \frac{1}{|B(x, \rho)|} \mathbf{1}_{B(x, \rho)}(y) dy$.

A particular family of Orlicz bump functions that is crucial for our theorem is the family

$$(1.6) \quad \Phi_m(t) = e^{\left((\ln t)^{\frac{1}{m}} + 1 \right)^m}, \quad t > E_m = e^{2^m}, \quad m > 1,$$

which is then extended in (7.19) below to be linear on the interval $[0, E_m]$ and submultiplicative on $[0, \infty)$, and which we discuss in more detail in Subsection 19.1.

DEFINITION 7. Let Ω be a bounded domain in \mathbb{R}^n . Fix $x \in \Omega$ and $\rho > 0$. We say ϕ is A -admissible at (x, ρ) if

$$\|\phi\|_{X(B(x, \rho))} \equiv \sup_{v \in (W_A^{1,1})_0(B(x, \rho))} \frac{\int_{B(x, \rho)} |v\phi| \, dy}{\int_{B(x, \rho)} \|\nabla_A v\| \, dy} < \infty.$$

Finally we recall that a measurable function u in Ω is *locally bounded above* at x if u can be modified on a set of measure zero so that the modified function \tilde{u} is bounded above in some neighbourhood of x .

THEOREM 8 (abstract local boundedness). Let Ω be a bounded domain in \mathbb{R}^n . Suppose that $\mathcal{A}(x, z)$ is a nonnegative semidefinite matrix in $\Omega \times \mathbb{R}$ that satisfies the structural condition (1.4). Let $d(x, y)$ be a symmetric metric in Ω , and suppose that $B(x, r) = \{y \in \Omega : d(x, y) < r\}$ with $x \in \Omega$ are the corresponding metric balls. Fix $x \in \Omega$. Then every weak subsolution of (1.3) is locally bounded above at x provided there is $r_0 > 0$ such that:

- (1) the function ϕ is A -admissible at (x, r_0) ,
- (2) the single scale (Φ, φ) -Sobolev Orlicz bump inequality (1.5) holds at (x, r_0) with $\Phi = \Phi_m$ for some $m > 2$,
- (3) there exists an (\mathcal{A}, d) -standard accumulating sequence of Lipschitz cutoff functions at (x, r_0) .

REMARK 9. The hypotheses required for local boundedness of weak solutions to $Lu = \phi$ at a single fixed point x in Ω are quite weak; namely we only need that the inhomogeneous term ϕ is A -admissible at **just one** point (x, r_0) for some $r_0 > 0$, and that there are two single scale conditions relating the geometry to the equation at **the one** point (x, r_0) .

REMARK 10. We could of course take the metric d to be the Carnot-Carathéodory metric associated with A , but the present formulation allows for additional flexibility in the choice of balls used for Moser iteration.

In the special case that a weak subsolution u to (1.3) is *nonpositive* on the boundary of a ball $B(x, r_0)$, we can obtain a global boundedness inequality $\|u\|_{L^\infty(B(x, r_0))} \lesssim \|\phi\|_{X(B(x, r_0))}$ from the arguments used for Theorem 8, simply by noting that integration by parts no longer requires premultiplication by a Lipschitz cutoff function. Moreover, the ensuing arguments work just as well for an arbitrary bounded open set Ω in place of the ball $B(x, r_0)$, provided only that we assume our Sobolev inequality for Ω instead of for the ball $B(x, r_0)$. Of course there is no role played here by a superradius φ . This type of result is usually referred to as a *maximum principle*, and we now formulate our theorem precisely.

DEFINITION 11. Fix a bounded domain $\Omega \subset \mathbb{R}^n$. Then the Φ -Sobolev Orlicz bump inequality for Ω is:

$$(1.7) \quad \Phi^{(-1)} \left(\int_{\Omega} \Phi(w) \, dx \right) \leq C \|\nabla_A w\|_{L^1(\Omega)}, \quad w \in Lip_0(\Omega),$$

where dx is Lebesgue measure in \mathbb{R}^n .

DEFINITION 12. Fix a bounded domain $\Omega \subset \mathbb{R}^n$. We say ϕ is A -admissible for Ω if

$$\|\phi\|_{X(\Omega)} \equiv \sup_{v \in (W_A^{1,1})_0(\Omega)} \frac{\int_{\Omega} |v\phi| \, dy}{\int_{\Omega} \|\nabla_A v\| \, dy} < \infty.$$

We say a function $u \in W_A^{1,2}(\Omega)$ is *bounded by a constant* $\ell \in \mathbb{R}$ on the boundary $\partial\Omega$ if $(u - \ell)^+ = \max\{u - \ell, 0\} \in \left(W_A^{1,2}\right)_0(\Omega)$. We define $\sup_{x \in \partial\Omega} u(x)$ to be $\inf\left\{\ell \in \mathbb{R} : (u - \ell)^+ \in \left(W_A^{1,2}\right)_0(\Omega)\right\}$.

THEOREM 13 (abstract maximum principle). *Let Ω be a bounded domain in \mathbb{R}^n . Suppose that $\mathcal{A}(x, z)$ is a nonnegative semidefinite matrix in $\Omega \times \mathbb{R}$ that satisfies the structural condition (1.4). Let u be a nonnegative subsolution of (9.2). Then the following maximum principle holds,*

$$\operatorname{esssup}_{x \in \Omega} u(x) \leq \sup_{x \in \partial\Omega} u(x) + C \|\phi\|_{X(\Omega)},$$

where the constant C depends only on Ω , provided that:

- (1) the function ϕ is A -admissible for Ω ,
- (2) the Φ -Sobolev Orlicz bump inequality (1.7) for Ω holds with $\Phi = \Phi_m$ for some $m > 2$.

In order to obtain a *geometric* local boundedness theorem, as well as a geometric maximum principle, we will take the metric d in Theorem 8 to be the Carnot-Caratheodory metric associated with the vector field ∇_A , and we will replace the hypotheses (2) and (3) in Theorem 8 with a geometric description of appropriate balls. For this we need to introduce a family of infinitely degenerate geometries that are simple enough that we can compute the balls, prove the required Sobolev Orlicz bump inequality, and define an appropriate accumulating sequence of Lipschitz cutoff functions. We will work initially in the plane and consider linear operators of the form

$$Lu(x, y) \equiv \nabla^{\operatorname{tr}} \mathcal{A}((x, y), u(x, y)) \nabla u(x, y), \quad (x, y) \in \Omega,$$

where $\Omega \subset \mathbb{R}^2$ is a planar domain, and where the 2×2 matrix $\mathcal{A}((x, y), z)$ is comparable to $A(x) = \begin{bmatrix} 1 & 0 \\ 0 & f(x)^2 \end{bmatrix}$, i.e. $\mathcal{A}((x, y), z)$ has bounded measurable coefficients satisfying

$$(1.8) \quad \frac{1}{C} \left(\xi^2 + f(x)^2 \eta^2 \right) \leq (\xi, \eta) A((x, y), z) \begin{pmatrix} \xi \\ \eta \end{pmatrix} \leq C \left(\xi^2 + f(x)^2 \eta^2 \right), \quad (x, y) \in \Omega, z \in \mathbb{R},$$

and where $f(x) = e^{-F(x)}$ is even and there is $R > 0$ such that F satisfies five structure conditions for some constants $C \geq 1$ and $\varepsilon > 0$:

DEFINITION 14 (structure conditions).

- (1) $\lim_{x \rightarrow 0^+} F(x) = +\infty$;
- (2) $F'(x) < 0$ and $F''(x) > 0$ for all $x \in (0, R)$;
- (3) $\frac{1}{C} |F'(r)| \leq |F'(x)| \leq C |F'(r)|$ for $\frac{1}{2}r < x < 2r < R$;
- (4) $\frac{1}{-xF'(x)}$ is increasing in the interval $(0, R)$ and satisfies $\frac{1}{-xF'(x)} \leq \frac{1}{\varepsilon}$ for $x \in (0, R)$;
- (5) $\frac{F''(x)}{-F'(x)} \approx \frac{1}{x}$ for $x \in (0, R)$.

REMARK 15. We make no smoothness assumption on f other than the existence of the second derivative f'' on the open interval $(0, R)$. Note also that at one extreme, f can be of finite type, namely $f(x) = x^\alpha$ for any $\alpha > 0$, and at the other extreme, f can be of strongly degenerate type, namely $f(x) = e^{-\frac{1}{x^\alpha}}$ for any $\alpha > 0$. Assumption (1) rules out the elliptic case $f(0) > 0$.

In the next two theorems we will consider the geometry of balls defined by

$$\begin{aligned} F_{k,\sigma}(r) &= \left(\ln \frac{1}{r} \right) \left(\ln^{(k)} \frac{1}{r} \right)^\sigma; \\ f_{k,\sigma}(r) &= e^{-F_{k,\sigma}(r)} = e^{-\left(\ln \frac{1}{r} \right) \left(\ln^{(k)} \frac{1}{r} \right)^\sigma}, \end{aligned}$$

where $k \in \mathbb{N}$ and $\sigma > 0$. Note that $f_{k,\sigma}$ vanishes to infinite order at $r = 0$, and that $f_{k,\sigma}$ vanishes to a faster order than $f_{k',\sigma'}$ if either $k < k'$ or if $k = k'$ and $\sigma > \sigma'$.

THEOREM 16 (geometric local boundedness). *Let $\Omega \subset \mathbb{R}^2$ and $\mathcal{A}(x, z)$ be a nonnegative semi-definite matrix in $\Omega \times \mathbb{R}$ that satisfies the structural condition (1.4), and assume in addition that $A(x) = \begin{bmatrix} 1 & 0 \\ 0 & f(x)^2 \end{bmatrix}$ where $f = f_{k,\sigma}$. Then every weak subsolution of (1.3) is locally bounded above in $\Omega \subset \mathbb{R}^2$ provided that:*

- (1) ϕ is A -admissible at $((0, y), r_y)$ for every y and some r_y depending on y , and
- (2) at least one of the following two conditions hold:
 - (a) $k \geq 1$ and $0 < \sigma < 1$,
 - (b) $k \geq 2$ and $\sigma > 0$.

THEOREM 17 (geometric maximum principle). *Let $\Omega \subset \mathbb{R}^2$ and $\mathcal{A}(x, z)$ be a nonnegative semi-definite matrix in $\Omega \times \mathbb{R}$ that satisfies the structural condition (1.4), and assume in addition that $A(x) = \begin{bmatrix} 1 & 0 \\ 0 & f(x)^2 \end{bmatrix}$ where $f = f_{k,\sigma}$. Let u be a subsolution of (9.2). Then we have the maximum principle,*

$$\operatorname{esssup}_{x \in \Omega} u(x) \leq \sup_{x \in \partial\Omega} u(x) + C \|\phi\|_{X(\Omega)},$$

provided that:

- (1) ϕ is A -admissible for Ω , and
- (2) at least one of the following two conditions hold:
 - (a) $k \geq 1$ and $0 < \sigma < 1$,
 - (b) $k \geq 2$ and $\sigma > 0$.

In Part 9 of the paper, we extend this result to hold in three dimensions, where we replace the inverse metric tensor in the plane $\begin{bmatrix} 1 & 0 \\ 0 & f(x_1)^2 \end{bmatrix}$, $f(s) = e^{-F(s)}$, with the analogous three dimensional matrix

$$A(x) \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & f(x_1)^2 \end{bmatrix}, \quad f(s) = e^{-F(s)},$$

and consider instead the operator

$$L_1 u(x, y) \equiv \nabla^{\operatorname{tr}} \mathcal{A}(x, u(x)) \nabla u(x), \quad x \in \Omega,$$

where $\Omega \subset \mathbb{R}^3$ and the 3×3 matrix $\mathcal{A}(x, z)$ is comparable to $A(x)$ above. Thus $\mathcal{A}((x, y), z)$ has bounded measurable coefficients satisfying

$$(1.9) \quad \frac{1}{C} \left(\xi_1^2 + \xi_2^2 + f(x_1)^2 \xi_3^2 \right) \leq (\xi_1, \xi_2, \xi_3) \mathcal{A}(x, z) \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} \leq C \left(\xi_1^2 + \xi_2^2 + f(x_1)^2 \xi_3^2 \right), \quad (x, y) \in \Omega, z \in \mathbb{R}.$$

THEOREM 18. *Let $\Omega \subset \mathbb{R}^3$. Suppose that u is a weak subsolution to the infinitely degenerate equation*

$$L_1 u \equiv \nabla^{\operatorname{tr}} \mathcal{A} \nabla u = \phi \text{ in } \Omega,$$

where the matrix $\mathcal{A}(x, z)$ satisfies (1.9), and where the degeneracy function f in (1.9) is comparable to $f_{k,\sigma}$. Then u is both locally bounded above in $\Omega \subset \mathbb{R}^2$, and satisfies the maximum principle,

$$\operatorname{esssup}_{x \in \Omega} u(x) \leq \sup_{x \in \partial\Omega} u(x) + C \|\phi\|_{X(\Omega)} ,$$

provided that:

- (1) ϕ is A -admissible for Ω , and
- (2) at least one of the following two conditions hold:
 - (a) $k \geq 1$ and $0 < \sigma < 1$,
 - (b) $k \geq 2$ and $\sigma > 0$.

1.1. Methods and techniques of proof. We restrict attention to the plane here for purposes of exposition. Since the quadratic form A is equal to a sum of squares of two Lipschitz vector fields $\frac{\partial}{\partial x}, f(x) \frac{\partial}{\partial y}$, the two standard notions of Sobolev space coincide, i.e. $W_A^{1,2} = H_A^{1,2}$ (see e.g. [FrSeSe], [GaNh] and [SaW3]). Thus the classical nondegenerate Sobolev space $W^{1,2}$ is dense in $W_A^{1,2}$, and we see that a classical $W^{1,2}$ -weak solution is also a $W_A^{1,2}$ -weak solution, thus granting the $W_A^{1,2}$ -weak solution the status as most general weak solution. Moreover, gradients in $W_A^{1,2}$ are unique and the usual calculus of gradients is at our disposal (see e.g. [SaW3]). Finally, we note that if u is a weak solution of $\mathcal{L}u = \nabla^{\text{tr}} \mathcal{A} \nabla u = 0$, then the classical Cacciopoli inequality, involving only integration by parts, shows that the L^2 norm of the degenerate form $\sqrt{\nabla u^{\text{tr}} \mathcal{A}((x, y), u(x, y))} \nabla u$ is controlled by the L^2 norm of the solution u . On the other hand, the inhomogeneous Sobolev Orlicz bump inequality for Lipschitz functions w , and the degenerate vector field

$$(1.10) \quad \nabla_A w \equiv \left(\frac{\partial w}{\partial x}, f(x) \frac{\partial w}{\partial y} \right),$$

requires special properties of the degeneracy function f . It is the equivalence of the L^2 norms of the degenerate form and the degenerate gradient, which is implied by (1.8), that permits the iteration of Moser.

Recall now that the method of Moser iteration plays off a Sobolev inequality, that holds for all functions, against a Cacciopoli inequality, that holds only for subsolutions or supersolutions of the linear equation. First, from results of Korobenko, Maldonado and Rios in [KoMaRi], it is known that if there exists a Sobolev bump inequality of the form

$$\|u\|_{L^q(\mu_B)} \leq Cr(B) \|\nabla_A u\|_{L^p(\mu_B)}, \quad u \in Lip_{\text{compact}}(B),$$

for some pair of exponents $1 \leq p < q \leq \infty$, and where the balls B are the Carnot-Carathéodory control balls for the degenerate vector field $\nabla_A = \left(\frac{\partial}{\partial x}, f \frac{\partial}{\partial y} \right)$ with radius $r(B)$, and $d\mu_B(x, y) = \frac{dx dy}{|B|}$ is normalized Lebesgue measure on B , then Lebesgue measure must be *doubling* on control balls, and so f cannot vanish to infinite order. Thus we must search for a weaker Sobolev bump inequality, and the natural setting for this is an inhomogeneous Sobolev Orlicz bump inequality

$$(1.11) \quad \Phi^{(-1)} \left(\int_B \Phi(|u|) d\mu_B \right) \leq C \varphi(r(B)) \|\nabla_A u\|_{L^1(\mu_B)}, \quad u \in Lip_{\text{compact}}(B),$$

where the function $\Phi(t)$ is increasing to ∞ and convex on $(0, \infty)$, but asymptotically closer to the identity t than any power function $t^{1+\sigma}$, $\sigma > 0$. The ‘superradius’ $\varphi(r)$ here is nondecreasing and $\varphi(r) \geq r$, and we show in Lemma 102 below that in certain cases where $\Phi(t)$ is closer to t on $(0, 1)$

than is any power $t^{1+\sigma}$, and where $\Phi'(0) = 0$, then $\lim_{r \rightarrow 0} \frac{\varphi(r)}{r} = \infty$. Note that an L^1 inequality such as (1.11) implies an L^2 version,

$$\Phi^{(-1)} \left(\int_B \Phi(|u|^2) d\mu_B \right) \leq C \left\{ \|u\|_{L^2(\mu_B)}^2 + \varphi(r(B))^2 \|\nabla_A u\|_{L^2(\mu_B)}^2 \right\}, \quad u \in Lip_{\text{compact}}(B),$$

which can then be used with Cacciopoli's inequality (see below) to control weak solutions. The left hand side above is not in general homogeneous in u , but this plays no role in the subsequent Moser iterations below, and in any event can be accounted for by rescaling u . Such a Sobolev inequality with a Φ -bump loses an entire degenerate derivative, but gains back a small amount Φ in integrability. We also point out that it will be important that Φ is sub (respectively super) multiplicative in the regions where t is large (respectively small), which necessitates choosing different 'formulas' for Φ in these two regions.

The other ingredient in Moser iteration is Cacciopoli's inequality that gains back the degenerate derivative, but only for *subsolutions* or *supersolutions* u of the equation $\mathcal{L}u = \phi$:

$$(1.12) \quad \int_B \psi_B^2 \|\nabla_A u\|^2 \leq C \int_B u^2 \|\nabla_A \psi_B\|^2,$$

where ψ_B is a smooth cutoff function adapted to the ball B , and where ϕ is A -admissible, i.e. $\|\phi\|_X < \infty$ (see Definition 7 above). Note again that the equivalence

$$\left| \frac{\partial u}{\partial x}(x, y) \right|^2 + f(x)^2 \left| \frac{\partial u}{\partial y}(x, y) \right|^2 = \|\nabla_A u(x, y)\|^2 \approx \nabla u(x, y)^{\text{tr}} \mathcal{A}((x, y), u(x, y)) \nabla u(x, y)$$

permits us to use Cacciopoli's inequality in conjunction with the Sobolev Orlicz bump inequality.

More precisely, in order to combine the Sobolev and Cacciopoli inequalities to provide an integrability gain for subsolutions that can be iterated, it suffices in some cases to assume that $u \geq \|\phi\|_X$ and that

- Φ is submultiplicative for t large, and
- $\sqrt{\Phi^{(n)}} \circ u^2$ is a subsolution of $Lu = \phi$ whenever u is a subsolution and $n \geq 0$.

Then we obtain a sequence of inequalities of the form

$$(1.13) \quad \Phi^{(-1)} \left(\frac{1}{\gamma_n} \int_{B(0, r_{n+1})} \Phi(|f_n(u)|^2) d\mu_{r_{n+1}} \right) \leq C_n \int_{B(0, r_n)} |f_n(u)|^2 d\mu_{r_n},$$

where the balls $B(0, r_n)$ shrink to a ball $B(0, r_\infty)$ with $r_\infty > 0$, whenever $f_n(u)$ is a subsolution of $\mathcal{L}u = \phi$. Now we assume that the function $\Phi(t)$ and the subsolution u satisfy the following two *key* inequalities:

$$(1.14) \quad \liminf_{n \rightarrow \infty} \left[\Phi^{(n)} \right]^{-1} \left(\int_{B(0, r_n)} \Phi^{(n)}(u^2) d\mu_{r_n} \right) \geq \|u\|_{L^\infty(\mu_{r_\infty})}^2,$$

and

$$(1.15) \quad \liminf_{n \rightarrow \infty} \left[\Phi^{(n)} \right]^{-1} \left(\Lambda^{(n)} \left(\|u\|_{L^2(\mu_{r_0})} \right) \right) \leq C(\|u\|_{L^2(\mu_{r_0})})$$

where $\Lambda^{(n)}$ is derived from iteration of Φ and the constants in (1.13). With these two key properties in hand we derive the *Inner Ball* inequality

$$(1.16) \quad \|u\|_{L^\infty(B_\infty)} \leq C \left(\|u\|_{L^2(B_0)} + \|\phi\|_{X(B_0)} \right),$$

which says that if u is a weak subsolution of $\mathcal{L}u = \phi$ on a ball B_0 , then u is a bounded function on a smaller ball B_∞ concentric with B . There is also a global version with both B_∞ and B_0 replaced by a single open set Ω , when in addition u vanishes in the weak sense on $\partial\Omega$:

$$(1.17) \quad \|u\|_{L^\infty(\Omega)} \leq C \|\phi\|_{X(\Omega)}.$$

It turns out that the first key property (1.14) is satisfied by essentially all of the Orlicz bump functions we consider, and so it is the second key property (1.15) that is decisive for the *Inner Ball* inequality (1.16) and its global counterpart (1.17). More precisely, when Φ is the Orlicz bump function Φ_m introduced in (1.6) above, the first key property (1.14) is satisfied for all $m > 1$, but the second key inequality (1.15) is only satisfied for $m > 2$. In fact, even if we take unreasonably small constants in the definition of $\Lambda^{(n)}$, the left hand side of (1.15) is **infinite** when $m = 2$, as is shown in Remark 46 below. This presents an obstacle to the use of Moser iteration in the absence of a Sobolev Orlicz inequality with bump function Φ_m for some $m > 2$, and ultimately accounts for the restriction to $k = 1$ and $\sigma < 1$ in the geometric local boundedness and maximum principle Theorems 16, 17 and 18. On the other hand, Theorem 115 in Part 9 provides a counterexample to the local boundedness assertion in Theorem 18 for the geometries $D_\sigma(r) = (\frac{1}{r})^\sigma$ when $\sigma > 1$. But for the intermediate geometries - namely $F_{1,\sigma}$ for $\sigma > 1$ and D_σ for $\sigma < 1$ - the question remains open as to whether or not we have local boundedness, or a maximum principle, for weak subsolutions. It is not clear at this point whether or not the above obstruction to the Moser method is the culprit. There may be counterexamples for (some of) these intermediate geometries, or there might be a different approach altogether which proves local boundedness and a maximum principle for (some of) these geometries.

It remains to obtain a sufficient condition for local boundedness that is based solely on the function $f(x)$ that measures geometric degeneracy of the equation. Given a nonnegative f that vanishes to infinite order at the origin, the result mentioned above in [KoMaRi] shows that the function $\Phi(t)$ must be asymptotically smaller than any power $t^{1+\sigma}$ with $\sigma > 0$ in order that the Sobolev Orlicz bump inequality (1.11) holds. On the other hand, $\Phi(t)$ must be asymptotically large enough that the Inner Ball inequality (1.16) holds for all subsolutions u to the equation $\mathcal{L}u = \phi$. As discussed in the previous paragraph, given the Sobolev Orlicz bump inequality (1.11) relative to the degeneracy function f , the Inner Ball inequality (1.16) is then *independent* of any further properties of f , and depends only on Φ . In fact, it holds ‘roughly speaking’ if and only if

$$(1.18) \quad \liminf_{t \rightarrow \infty} \frac{\Phi(t)}{t^{1+\frac{1}{\sqrt{\ln t}}}} = \infty,$$

ie. $\Phi(L)$ is at least as large as $Le^{\sqrt{\log L}}$, which is asymptotically much larger than $L \log L$. A natural family of bump functions $\bar{\Phi}_m(t)$ to consider in regard to (1.18) is given by

$$\bar{\Phi}_m(t) = t^{1+\frac{m}{(\ln t)^{\frac{1}{m}}}},$$

but a significant drawback to this family is that the iterations $\bar{\Phi}_m^{(n)}$ appearing in (1.14) and (1.15) above are extremely difficult to estimate appropriately. An essentially comparable, but far more convenient, family of bump functions Φ_m is the family introduced in (1.6) above, and given by

$$\Phi_m(t) = e^{\left[(\ln t)^{\frac{1}{m}} + 1\right]^m},$$

where $\ln \Phi_m(t) \approx \ln \bar{\Phi}_m(t)$, and the iterations $\Phi_m^{(n)}$ are trivially given by $\Phi_m^{(n)}(t) = e^{\left[(\ln t)^{\frac{1}{m}} + n\right]^m}$. For such an Orlicz bump function Φ_m with $m > 2$, it then turns out that ‘roughly speaking’, the

Sobolev Orlicz bump inequality (1.11) holds relative to the degeneracy function f , if and only if

$$(1.19) \quad \liminf_{r \rightarrow 0} \frac{f(r)}{r^{(\ln \frac{1}{r})}} = \infty.$$

Thus we conclude that if f is ‘roughly speaking’ asymptotically greater than $r^{(\ln \frac{1}{r})}$ as $r \rightarrow 0$, then subsolutions to $\mathcal{L}u = \phi$ are locally bounded.

An actual counterexample to a local boundedness theorem for the homogeneous equation is presented in Part 9 of the paper, where we consider the extension

$$\mathcal{L}_1 \equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + f(x_1)^2 \frac{\partial^2}{\partial x_3^2}$$

of \mathcal{L} to three dimensions. This extension is ‘more degenerate’ than \mathcal{L} due to the ‘larger vanishing set’ of the function $f(x_1)$. Kusuoka and Strook [KuStr] have shown that when $f(x_1)$ is smooth and positive away from the origin, the smooth linear operator \mathcal{L}_1 is hypoelliptic *if and only if*

$$\lim_{r \rightarrow 0} r \ln f(r) = 0.$$

In part (1) of Theorem 114, and in Theorem 115 below, we consider local boundedness of weak solutions to rough divergence form operators $L_1 = \operatorname{div} \mathcal{A} \nabla$ with quadratic forms \mathcal{A} controlled by that of \mathcal{L}_1 , and demonstrate that for $f \approx f_{k,\sigma}$ and ϕ admissible:

- weak solutions u to $L_1 u = \phi$ are locally bounded if

$$\text{either } k \geq 2 \text{ and } \sigma > 0; \text{ or } k = 1 \text{ and } 0 < \sigma < 1;$$

- there exist unbounded weak solutions u to the homogeneous equation $L_1 u = 0$ if

$$k = 0 \text{ and } \sigma \geq 1.$$

The range of degeneracy parameters for which we obtain unbounded weak solutions to rough divergence form operators L_1 thus coincides with the range where the smooth operator \mathcal{L}_1 fails to be hypoelliptic.

PROBLEM 19 (Moser Gap). *Are all weak subsolutions to an admissible equation locally bounded when*

- (1) *the equation is $Lu = \phi$ in the plane with geometry $F_{1,\sigma}$ for $\sigma \geq 1$ or with geometry $F_{0,\sigma}$ for $\sigma > 0$?*
- (2) *the equation is $L_1 u = \phi$ in \mathbb{R}^3 with geometry $F_{1,\sigma}$ for $\sigma \geq 1$ or with geometry $F_{0,\sigma}$ for $0 < \sigma < 1$?*

2. Bombieri and DeGiorgi iteration and continuity of solutions

Now we turn to the question of obtaining continuity of weak solutions at a single point x to the equation $\mathcal{L}u = \phi$. Let Ω be a bounded domain in \mathbb{R}^n and recall the quadruple $(\mathcal{A}, d, \varphi, \Phi)$ of objects of interest we introduced above. For continuity of solutions we need to assume stronger connections between these objects. For example we will need to assume the three conditions in Theorem 8, but over **all** scales r satisfying $0 < r \leq r_0$, for some $r_0 > 0$. We will also need further strengthenings, beginning with the concept of ‘doubling increment’ of a ball, and its connection with the superradius.

DEFINITION 20. Let Ω be a bounded domain in \mathbb{R}^n . Let $\delta_x(r)$ be defined implicitly by

$$(1.20) \quad |B(x, r - \delta(r))| = \frac{1}{2} |B(x, r)|,$$

We refer to $\delta_x(r)$ as the doubling increment of the ball $B(x, r)$.

CONDITION 21 (Doubling increment growth condition). Let Ω be a bounded domain in \mathbb{R}^n . Let $\delta_x(r)$ be the doubling increment of $B(0, r)$ defined by (1.20), and let $\varphi(r)$ be the superradius as in (1.11). We say that $\delta_x(r)$ satisfies the doubling increment growth condition if for any $\varepsilon > 0$ there exists $r_\varepsilon > 0$ such that

$$(1.21) \quad \left(\ln \frac{\varphi(r)}{\delta_x(r)} \right)^m \leq \varepsilon \ln^{(3)} 1/r, \quad \forall r \leq r_\varepsilon.$$

DEFINITION 22 (Nonstandard sequence of accumulating Lipschitz functions). Let Ω be a bounded domain in \mathbb{R}^n . Let $r > 0$, $x \in \Omega$ and define an (\mathcal{A}, d) -nonstandard sequence of Lipschitz cutoff functions $\{\psi_j\}_{j=1}^\infty$ at (x, r) , along with the sets $B(x, r_j) \supset \text{supp } \psi_j$ by setting $r_1 = r$,

$$r_{j+1} = r_j - \frac{1-\nu}{j^2} \delta_x(r_j),$$

and then

$$(1.22) \quad \begin{cases} B_1 &= \text{supp}(\psi_1) \subset \overline{B(x, r)}, \\ B(x, \nu r) &\subset \{y : \psi_j(y) = 1\}, \quad j \geq 1, \\ B_{j+1} = \text{supp}(\psi_{j+1}) &\Subset \{y : \psi_j(y) = 1\}, \quad j \geq 1, \\ \frac{|B_j|}{|B_{j+1}|} &\leq D, \quad j \geq 1, \\ |||\nabla_A \psi_j|||_{L^\infty(B(0, r))} &\leq \frac{Gj^2}{(1-\nu)\delta_x(r_j)}, \quad j \geq 1, \end{cases}$$

where $\delta_x(r)$ is defined implicitly by (1.20).

We will need to assume the previous single scale (Φ, φ) -Sobolev Orlicz bump inequality for an additional particular family, and also a 1-1 Poincaré inequality. The additional family of Orlicz bump functions that is crucial for our continuity theorem is the family

$$\begin{aligned} \Psi_m(t) &= A_m e^{-\left((\ln \frac{1}{t})^{\frac{1}{m}} + 1\right)^m}, \quad 0 < t < \frac{1}{M}; \\ A_m &= e^{((\ln M)^{1/m} + 1)^m - \ln M} > 1, \end{aligned}$$

where M is appropriately defined, and then Ψ_m is extended to be affine on the interval $[\frac{1}{M}, \infty)$ with slope $\Psi'_m(\frac{1}{M})$. Note that the (1,1) Poincaré inequality below holds with the usual radius in place of a superradius.

DEFINITION 23. Let Ω be a bounded domain in \mathbb{R}^n . Fix $x \in \Omega$ and $\rho > 0$. Then the single scale Poincaré inequality at (x, ρ) is:

$$(1.23) \quad \int_{B(x, \rho)} \left| g - \int_{B(x, \rho)} g d\mu_{x, \rho} \right| d\mu_{x, \rho} \leq C\rho \|\nabla_A g\|_{L^1(\mu_{x, \rho})}.$$

Here is the strengthening of the admissibility condition that we need for continuity.

DEFINITION 24. Let Ω be a bounded domain in \mathbb{R}^n . Let $\rho > 0$, $x \in \Omega$. We say ϕ is Dini A -admissible at (x, ρ) if ϕ is A -admissible at (x, ρ) , and if in addition, for every $0 < \tau < 1$,

$$\sum_{k=0}^{\infty} \|\phi\|_{X(B(y, \tau^k \rho))} < \infty.$$

Finally we recall that a measurable function u in Ω is *continuous* at x if u can be modified on a set of measure zero so that the modified function \tilde{u} is continuous at the point x .

THEOREM 25 (abstract continuity). Let Ω be a bounded domain in \mathbb{R}^n . Suppose that $\mathcal{A}(x, z)$ is a nonnegative semidefinite matrix in $\Omega \times \mathbb{R}$ and satisfies (1.4). Let $d(x, y)$ be a symmetric metric in Ω , and suppose that $B(x, r) = \{y \in \Omega : d(x, y) < r\}$ with $x \in \Omega$ are the corresponding metric balls. Fix $x \in \Omega$. Then every weak solution of (1.3) is continuous at x provided there is an increasing function $\varphi : (0, 1) \rightarrow (0, \infty)$ with $\varphi(r) \geq r$, and a positive number $r_0 > 0$ such that:

- (1) the function ϕ is Dini A -admissible at (x, r) for all $0 < r \leq r_0$,
- (2) the (Φ, φ) -Sobolev Orlicz bump inequality (1.5) holds at (x, r) for all $0 < r \leq r_0$, with (a) $\Phi = \Phi_m$ for some $m > 2$ and also with (b) $\Phi = \Psi_m$ for some $m > 2$,
- (3) the 1-1 Poincaré inequality (1.23) holds at (x, r) for all $0 < r \leq r_0$,
- (4) there exists a nonstandard accumulating sequence of Lipschitz cutoff functions at (x, r) for all $0 < r \leq r_0$,
- (5) the doubling increment $\delta_x(r)$ satisfies the doubling increment growth Condition 21 with superradius $\varphi(r)$.

Our corresponding geometric theorems for continuity in two and three dimensions are these.

THEOREM 26 (geometric continuity). Let $\Omega \subset \mathbb{R}^2$ and $\mathcal{A}((x, y), z)$ be a nonnegative semidefinite matrix in $\Omega \times \mathbb{R}$ that satisfies (1.4), and assume in addition that $A(x) = \begin{bmatrix} 1 & 0 \\ 0 & f(x)^2 \end{bmatrix}$ where $f = f_{k, \sigma}$. Then every weak solution of (1.3) is continuous in $\Omega \subset \mathbb{R}^2$ provided

- (1) either $k \geq 4$ and $\sigma > 0$, or $k = 3$ and $0 < \sigma < 1$,
- (2) and ϕ is Dini A -admissible where the balls $B(x, r)$ in Definition 24 are taken with respect to the Carnot-Carathéodory metric $d(x, y)$ associated with $A(x)$.

THEOREM 27. Let $\Omega \subset \mathbb{R}^3$ and $\mathcal{A}(x, z)$ be a nonnegative semidefinite matrix in $\Omega \times \mathbb{R}$ that satisfies (1.9), and assume in addition that $A(x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & f(x_1)^2 \end{bmatrix}$ where $f = f_{k, \sigma}$. Then every weak solution u to the infinitely degenerate equation

$$L_1 u \equiv \nabla^{\text{tr}} \mathcal{A} \nabla u = \phi \text{ in } \Omega \subset \mathbb{R}^3,$$

is continuous in Ω provided (1) either $k \geq 4$ and $\sigma > 0$, or $k = 3$ and $0 < \sigma < \frac{1}{m-1}$, and (2) provided ϕ is Dini A -admissible where the balls $B(x, r)$ in Definition 24 are taken with respect to the Carnot-Carathéodory metric $d(x, y)$ associated with $A(x)$.

In our arguments below that prove continuity of weak solutions, we will need to establish a number of different Inner Ball inequalities, each requiring a different Cacciopoli inequality. In particular, the Cacciopoli inequality in Section 3, that is crucial for deriving continuity of weak solutions, is necessarily weaker than the standard inequality (1.12), and poses additional obstacles.

2.1. Methods and techniques of proof. Since we may assume that our weak solutions are now bounded by taking both f and $-f$ as in Theorem 16 above, we need to carefully define our bump function $\Phi(t)$ for t small, rather than for t large as in the derivation of local boundedness of weak subsolutions above.

The basic idea in Bombieri iteration is to perform a sequence of *generalized* Moser iterations between consecutive balls $B(0, \nu_{j+1})$ with $0 < \nu_j < \nu_{j+1} \nearrow 1$. The *generalized* Inner Ball inequalities used here are rescalings of a previous Inner Ball inequality and have the form

$$(1.24) \quad \|h(u)\|_{L^\infty(\nu B_r)} \leq C_r e^{c(\ln \frac{1}{1-\nu})^m} \|h(u)\|_{L^2(B_r)},$$

where $h(u)$ is a nonlinear function of a weak solution u and $h(t)$ is ‘close’ to either $\ln t$ or $\ln \frac{1}{t}$. Then from an appropriate application of Bombieri’s iteration to these generalized Inner Ball inequalities, we obtain a Harnack inequality for positive solutions to $\mathcal{L}u = \phi$ of the form:

$$\operatorname{esssup}_{x \in B(y, \nu r)} \left(u(x) + \|\phi\|_{X(B(y, r))} \right) \leq C_{Har}(y, r, \nu) \operatorname{essinf}_{x \in B(y, \nu r)} \left(u(x) + \|\phi\|_{X(B(y, r))} \right).$$

Now it turns out that if the Harnack constant $C_{Har}(r)$ satisfies

$$C_{Har}(r) \leq C \left(\ln \frac{1}{r} \right) \left(\ln \ln \frac{1}{r} \right), \quad r \ll 1,$$

then $\sum_k \frac{1}{C_{Har}(\tau^k)} = \infty$ and a well known clever iteration argument of DeGiorgi yields continuity of weak solutions. If however,

$$C_{Har}(r) \geq C \left(\ln \frac{1}{r} \right) \left(\ln \ln \frac{1}{r} \right)^{1+\varepsilon}, \quad r \ll 1,$$

then $\sum_k \frac{1}{C_{Har}(\tau^k)} < \infty$ and the *method* fails to yield continuity of weak solutions.

We show below that for an Orlicz bump function Φ satisfying (1.18), and in particular the degeneracy function

$$f(r) \approx f_{k, \sigma}(r) = r^{(\ln^{(k)} \frac{1}{r})^\sigma}, \quad k \geq 2, \sigma > 0,$$

the Orlicz bump inequality (1.11) holds and $C_{Har}(r)$ is finite. However, the Harnack constant $C_{Har}(r)$ depends in a complicated way on f , but it turns out that ‘roughly speaking’ we have $C_{Har}(r) \leq C \left(\ln \frac{1}{r} \right) \left(\ln \ln \frac{1}{r} \right)$ provided

$$f(r) \approx f_{k, \sigma}(r), \quad \text{for small } r > 0, \text{ and either for some } k \geq 4 \text{ and } \sigma > 0, \\ \text{or for } k = 3 \text{ and } 0 < \sigma < 1,$$

and then all weak solutions to $\mathcal{L}u = \phi$ are continuous if in addition ϕ is Dini A -admissible.

Comparing this inequality to the rough condition $\liminf_{r \rightarrow 0} \frac{f(r)}{r^{(\ln \frac{1}{r})}} = \infty$, that is required for local boundedness, we see that in order to obtain continuity of weak solutions to $\mathcal{L}u = \phi$, we need our degeneracy function f to be much closer to finite type than required for local boundedness of subsolutions to $\mathcal{L}u = \phi$, namely two extra iterations of the logarithm in the exponent.

CHAPTER 2

The main new ideas and organization of the paper

There are at least four significant difficulties to be overcome in the infinitely degenerate regime, and these arise in establishing the Inner Ball inequality, the Sobolev Orlicz bump inequality, the new Cacciopoli inequalities, and the geometric estimates for the infinitely degenerate balls. We first describe these difficulties, and indicate how they are overcome, before turning our attention to the organization of the paper.

In order to prove the Inner Ball inequality (1.16) from the Orlicz bump inequality (1.11) and the appropriate Cacciopoli inequalities, we must establish (1.15) and (1.14) by a sequence of delicate recursive estimates using a special recursive form of Φ , namely

$$\Phi(t) = e^{g^{-1}(g(\ln t)+1)},$$

where the generator $g(s)$ satisfies

$$\begin{aligned} g'(x) &> 0 \text{ and decreasing for } x \text{ large,} \\ \lim_{x \rightarrow \infty} g(x) &= \infty \text{ and } \lim_{x \rightarrow \infty} \frac{g(x)}{x} = 0, \\ \sum_{j=N}^{\infty} \{g(g^{-1}(j)+1) - j\} \ln(j) &< \infty \text{ for some } N. \end{aligned}$$

Such representations of a given Φ are not unique, and for the examples of Φ that are close to the boundary of condition (1.18), suitable representations can be easily guessed.

In order to prove the Sobolev Orlicz bump inequality (1.11) for a given quadruple $(\mathcal{A}, d, \varphi, \Phi)$, we begin by establishing a subrepresentation inequality of the form

$$w(x) \leq C \int_{\Gamma(x,r)} |\nabla_A w(y)| K_{B(0,r)}(x, y) dy, \quad x \in B(0, r),$$

where the kernel $K_{B(0,r)}$ is given by

$$K_{B(0,r)}(x, y) = \frac{\widehat{d}(x, y)}{|B(x, d(x, y))|},$$

and where $d(x, y)$ is the control metric associated with $A(x)$, $B(x, r)$ is the associated ball and $|B(x, r)|$ is its Lebesgue measure. The set $\Gamma(x, r)$ is a degenerate ‘cusp’ centered at x . The novel feature here is that $\widehat{d}(x, y)$ is in general *much smaller* than the distance $d(x, y)$ when the metric $A(x)$ is infinitely degenerate, in fact it is given by

$$\widehat{d}(x, y) \equiv \min \left\{ d(x, y), \frac{1}{|F'(x_1 + d(x, y))|} \right\},$$

where $F = -\ln f$ is as above. Then straightforward arguments, but complicated by the necessity of using the superradius φ , are used to calculate the Sobolev Orlicz bump inequality (1.11). We also

give an example of a degenerate geometry $f(r) = e^{-\frac{1}{r}}$ for which the ‘classical’ subrepresentation inequality with kernel $\frac{d(x,y)}{|B(x,d(x,y))|}$ associated with this geometry *fails* to satisfy the $(1,1)$ -Poincaré estimate, while our subrepresentation inequality with kernel $\frac{\hat{d}(x,y)}{|B(x,d(x,y))|}$ easily recovers the $(1,1)$ -Poincaré estimate.

In order to obtain the appropriate Cacciopoli inequalities (1.12), we apply integrations by part as usual to iterations $h^{(n)}$ of a nonlinear convex function h (convexity is needed for Jensen’s inequality) composed with a weak solution u , or sometimes with a small positive power u^ε of u . In the classical subelliptic case, we can take $h(t) = t^\sigma$ for some $\sigma > 1$ determined by the subelliptic Sobolev embedding, and then for each $n \geq 1$, the function $h^{(n)}(t^\varepsilon) = t^{\sigma^n \varepsilon}$ is *either* strictly convex on $(0, \infty)$, *or* it is strictly concave on $(0, \infty)$. In either situation a Cacciopoli inequality can be obtained for weak (sub/super respectively) solutions. However, if $h(t)$ is instead taken to be a convex Young function $\Phi(t)$ for which a degenerate Sobolev embedding holds, then it is no longer necessarily the case that for a given $n \geq 1$, the function $h^{(n)}(t^\varepsilon) = \Phi^{(n)}(t^\varepsilon)$ is either convex on $(0, \infty)$ or concave on $(0, \infty)$ - instead its second derivative may change sign ‘uncontrollably’ often. This causes great difficulty in obtaining the weak Harnack inequality for small values of u , and requires further new ideas - see Section 3 below.

Finally, to prove the geometric properties needed for the control balls associated with the degeneracy function f , we use calculus of variation arguments to determine the geodesics, their arc lengths, and the areas of the control balls. These estimates are then used to derive the above subrepresentation formula. It might be useful for the reader to keep in mind the following scale of degenerate geometries parameterized by the function $F = \ln f$:

$$\begin{aligned} D_\sigma(r) &\equiv \left(\frac{1}{r}\right)^\sigma, & \sigma > 0, \\ F_{k,\sigma}(r) &\equiv \left(\ln \frac{1}{r}\right) \left(\ln^{(k)} \frac{1}{r}\right)^\sigma, & k \geq 1, \sigma > 0, \\ H_N(r) &\equiv N \ln \frac{1}{r}, & N \geq 0, \end{aligned}$$

that satisfy

$$H_{N_1}(r) \lesssim H_{N_2}(r) \lesssim F_{k_1,\sigma_1}(r) \lesssim F_{k_1,\sigma_2}(r) \lesssim F_{k_2,\sigma_1}(r) \lesssim F_{k_2,\sigma_2}(r) \lesssim D_{\sigma_1}(r) \lesssim D_{\sigma_2},$$

provided $N_1 \leq N_2$ and $k_1 \geq k_2$ and $\sigma_1 \leq \sigma_2$. Thus the smallest geometry $H_0(r) = 0$ corresponds to the elliptic Euclidean geometry, $H_N(r)$ corresponds to the finite type N geometries, $F_{k,\sigma}(r)$ corresponds to a near finite type geometry that drifts further from finite type as k decreases and σ increases, and $D_\sigma(r)$ corresponds to a very degenerate geometry whose degeneracy increases with σ . Note that if we formally set $k = 0$ in the definition of $F_{k,\sigma}$ we obtain

$$F_{0,\sigma}(r) = \left(\ln \frac{1}{r}\right) \left(\frac{1}{r}\right)^\sigma = \left(\ln \frac{1}{r}\right) D_\sigma(r) \approx D_\sigma(r).$$

Finally, we note that for the derivation of continuity, a complication arises in that we need to define $\Phi(t)$ for t small, with the consequence that Φ must now be *super*multiplicative rather than *sub*multiplicative - see Lemma 94 below and the discussion thereafter. This limits the type of arguments at our disposal.

1. Organization of the paper

The remainder of the paper is organized into nine more parts.

Part 2 is dedicated to the abstract parts of the theory, those that assume appropriate Sobolev and Poincaré inequalities hold, and then deduce properties of solutions. In Chapter 3 we prove Cacciopoli inequalities for solutions u , both for convex bumps $\Phi^{(n)}(u^\beta)$, $n \in \mathbb{N}$, when u is large and $\beta < 0$ or $\beta \geq 1$, and for convex and concave bumps $\Psi^{(k)}(u)$, $k \in \mathbb{Z}$, when u is small. Chapter 4 is dedicated to the local boundedness and maximum principle for weak subsolutions, and the proofs of Theorems 8 and 13. In Chapter 5 we consider Harnack inequalities and determine their constants in terms of Orlicz Sobolev inequalities in Theorem 51. This is then used to obtain continuity of weak solutions for the geometries f in Theorem 26. Theorem 1 in the introduction is then a corollary of Theorem 26 and the main result in [RSaW2], which we now reproduce here.

THEOREM 28 (Rios, Sawyer and Wheeden). *Let Ω be a strictly convex domain in \mathbb{R}^n containing the origin. Let $k^i(x, z)$, $i = 2, \dots, n$, be smooth nonnegative functions in $\Omega \times \mathbb{R}$ such that*

$$k^i(x, z) > 0 \quad \text{if } x_j \neq 0 \quad \text{for some } j \neq i$$

(this means that $k^i(x, z)$ may vanish only for those (x, z) so that x lies on the i^{th} -coordinate axis), and such that

$$\left| \frac{\partial}{\partial z} k^i(x, z) \right| = o(k^*(x, z)) \quad \text{as } z \rightarrow 0, \quad \text{for all } (x, z) \in \Omega \times \mathbb{R},$$

where $k^ = \min_{i=2, \dots, n} k^i$. Then, for any continuous function φ on $\partial\Omega$, there exists a unique continuous strong solution w to the Dirichlet problem*

$$\begin{cases} \frac{\partial^2}{\partial x_1^2} w(x) + \sum_{i=2}^n \frac{\partial}{\partial x_i} \left(k^i(x, w(x)) \frac{\partial}{\partial x_i} w(x) \right) = 0 & \text{in } \Omega, \\ w = \varphi & \text{on } \partial\Omega, \end{cases}$$

i.e., there exists a unique w that is both a strong solution of the differential equation in Ω and continuous in $\overline{\Omega}$ with boundary values φ . Moreover, this solution $w \in C^0(\overline{\Omega}) \cap C^\infty(\Omega)$.

Indeed, with $\mathcal{L}_{\text{quasi}} u \equiv \left\{ \frac{\partial^2 u}{\partial x^2} + f(x, u(x))^2 \frac{\partial^2 u}{\partial y^2} \right\}$ as in Theorem 1, we take $n = 2$ and $k^2((x_1, x_2), z) = f(x_1, z)^2$ in Theorem 28. The continuity of the weak solution w that is needed in Theorem 28 is guaranteed by the conclusion of Theorem 26.

Part 3 is concerned with geometry and its implications for the abstract theory in Part 2. In Chapter 6 we investigate the geometry of control balls in the plane \mathbb{R}^2 associated with f , and in particular compute the length of geodesics and the areas of balls. Then in Chapter 7 we use these geometric estimates to derive a sharp subrepresentation formula in the plane for functions in terms of the degenerate gradient, which is then used to prove a $(1, 1)$ -Poincaré inequality and the Sobolev Orlicz bump inequalities for triples (f, φ, Φ) where Φ is near optimal and where f essentially satisfies (1.19). It turns out that the situations where the function values are large or small are handled quite differently. Finally in Chapter 8 we derive the geometric versions of our local boundedness, maximum principle, and continuity theorems in the plane.

Part 4 is devoted to sharpness considerations. In Chapter 9 we discuss the inhomogeneous equation $\mathcal{L}u = \phi$ in more detail, and show that admissibility of ϕ is essentially necessary for the local boundedness and maximum principle for weak subsolutions u to $\mathcal{L}u = \phi$. Then in Chapter 10 we discuss an extension of our results to divergence form operators L_1 whose quadratic forms are comparable to that of

$$\mathcal{L}_1 \equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + f(x_1)^2 \frac{\partial^2}{\partial x_3^2},$$

and as mentioned earlier, we can here obtain an actual counterexample to the local boundedness theorem for solutions to the homogeneous equation. Namely, if $f(r) \leq Ce^{-\frac{1}{r}}$ for $0 < r < R$, then we show that there exist *unbounded* $W_A^{1,2}$ -weak solutions u to $\mathcal{L}_1 u = 0$. Note that this very degenerate geometry lies in the scale $D_\sigma(r)$, which is essentially the scale $F_{0,\sigma}(r) \equiv (\ln \frac{1}{r}) (\frac{1}{r})^\sigma$.

In the Appendix in Part 5 we collect three results tangential to the developments above. We show that our hypoellipticity result in Theorem 1 does *not* extend to equations with a stronger nonlinearity, such as the Monge-Ampère equation, even with an *arbitrary* infinite degeneracy. This, despite the close connection between the two dimensional Monge-Ampère equation and quasilinear equations that is exhibited by the partial Legendre transform. Then we show that generic Young functions have a recursive form that permits easy calculation of their iterates. Finally, we compute the Fedii operator \mathcal{L} in metric polar coordinates and show that there are no nonconstant radial functions u such that $\mathcal{L}u$ is radial, unlike in the elliptic case where radial solutions $u(r)$ need only satisfy $u''(r) + \frac{1}{r}u(r) = 0$.

Part 2

Abstract theory in higher dimensions

In this second part of the paper we prove local boundedness and maximum principles for weak subsolutions, and continuity for weak solutions, under the assumption that certain degenerate Orlicz Sobolev and Poincaré inequalities hold. This abstract theory holds in the greater generality of n -dimensional Euclidean space \mathbb{R}^n . In the first chapter here we *prove* Cacciopoli inequalities, and then in the next two chapters we use these inequalities, together with some *assumed* Sobolev and Poincaré inequalities, to treat local boundedness, the maximum principle and continuity of weak solutions.

CHAPTER 3

Cacciopoli inequalities for weak sub and super solutions u

Here in this chapter we introduce the notion of weak sub and super solutions and establish various Cacciopoli inequalities. Recall the definition of a classical weak (sub, super) solution.

DEFINITION 29. A function $u \in W_A^{1,2}(\Omega)$ is a weak $\begin{pmatrix} \text{solution} \\ \text{subsolution} \\ \text{supersolution} \end{pmatrix}$ of

$$(3.1) \quad \mathcal{L}u = \phi$$

in Ω , where $\phi \in L^2(\Omega)$, if

$$(3.2) \quad - \int (\nabla w)^{tr} A \nabla u \begin{pmatrix} = \\ \geq \\ \leq \end{pmatrix} \int \phi w,$$

for all nonnegative $w \in (W_A^{1,2})_0(\Omega)$.

In the case of a weak solution, we may equivalently test (3.2) over all $w \in (W_A^{1,2})_0(\Omega)$. Note that the integrals in (3.2) converge absolutely since $u \in W_A^{1,2}(\Omega)$ and $w \in (W_A^{1,2})_0(\Omega) \subset L^2(\Omega)$. In order to prove a Cacciopoli inequality, we assume that the inhomogeneous term ϕ in (3.1) is admissible for A in the sense of Definition 7, but applied globally as in the following variant.

DEFINITION 30. We say ϕ is A -admissible in an open set Ω if for every $y \in \Omega$ there exists $R_0 = R_0(y)$ such that

$$\|\phi\|_{X(B(y, R_0))} \equiv \sup_{v \in (W_A^{1,1})_0(B(y, R_0))} \frac{\int_{B(y, R_0)} |v\phi| \, dx}{\int_{B(y, R_0)} \|\nabla_A v\| \, dx} < \infty.$$

We say ϕ is Dini A -admissible in an open set Ω if in addition, for every $y \in \Omega$ and $0 < \tau < 1$ there exists $R_0 = R_0(y, \tau)$ such that

$$\sum_{k=0}^{\infty} \|\phi\|_{X(B(y, \tau^k R_0))} < \infty.$$

We will need to assume that ϕ is A -admissible in order to derive local boundness of weak subsolutions, and we will need to assume that ϕ is Dini A -admissible in order to derive continuity of weak solutions by applying De Giori iteration. Dini A -admissible functions ϕ arise naturally in Orlicz spaces $L^{\tilde{\Phi}}$ where $\tilde{\Phi}$ is the conjugate Young function to Φ - see Section 3 below for the definition of a conjugate Young function.

LEMMA 31. *A function ϕ is Dini A -admissible if there is a bump function Φ^* such that the Sobolev inequality holds for the control geometry associated with A , and such that $\phi \in L^{\Phi^*}$ where $\widetilde{\Phi}^*$ is the conjugate Young function to Φ^* .*

PROOF. We have $\|\phi\|_{X(B(y, R_0))} \lesssim R_0 \|\phi\|_{L^{\widetilde{\Phi}^*}(B(y, R_0))}$ since

$$\int_{B(y, R_0)} |v\phi| \, dx \leq \|v\|_{L^{\Phi^*}(B(y, R_0))} \|\phi\|_{L^{\widetilde{\Phi}^*}(B(y, R_0))} \leq C_1(B(y, R_0)) R_0 \|\nabla_A v\|_{L^1(B(y, R_0))} \|\phi\|_{L^{\widetilde{\Phi}^*}(B(y, R_0))},$$

where $C_1(\Omega)$ is the norm of the Φ^* -Sobolev inequality on $B(y, R_0)$. Moreover,

$$\begin{aligned} \sum_{k=0}^{\infty} \|\phi\|_{X(B(y, \tau^k R_0))} &\leq \sum_{k=0}^{\infty} \tau^k R_0 \|\phi\|_{L^{\widetilde{\Phi}^*}(B(y, \tau^k R_0))} \\ &\leq \sum_{k=0}^{\infty} \tau^k R_0 \|\phi\|_{L^{\widetilde{\Phi}^*}(B(y, R_0))} = C_\tau R_0 \|\phi\|_{L^{\widetilde{\Phi}^*}(B(y, R_0))} < \infty. \end{aligned}$$

■

Note that the larger the bump function Φ we can take in the Sobolev inequality, the larger the space $L^{\widetilde{\Phi}}$ we can take for ϕ . Here we emphasize that the Orlicz bump function Φ^* that bears witness to the admissibility of ϕ need not coincide with the bump function Φ . For the purpose of establishing a sharpness result later, we also define a stronger notion of admissibility.

DEFINITION 32. *We say ϕ is strongly A -admissible in an open set Ω if there is a bump function Φ^* such that the Sobolev inequality holds for the control geometry associated with A , and such that $\phi \in L^{\widetilde{\Phi}^*}$ where $\widetilde{\Phi}^*$ is the conjugate Young function to Φ^* .*

We will need Cacciopoli inequalities in three different situations, namely for large subsolutions u that arise in local boundedness, for small solutions u that arise in Bombieri inequalities for u^β as $\beta \nearrow 0$, and for small solutions u that arise in Bombieri inequalities for $\Psi^{(-N)}(u)$ as $N \nearrow \infty$. We now establish these inequalities in the next three sections.

1. Sub solutions of the form $\Gamma^{(n)}(u)$ with sub solution $u > M$

We begin by first establishing a reverse Sobolev inequality of Cacciopoli type for weak (sub, super) solutions u to $\mathcal{L}u = \phi$ where $\mathcal{L} = \nabla^{\text{tr}} A(x) \nabla$ and $A(x)$ is a bounded positive semidefinite matrix. Now let $u \in W_A^{1,2}(\Omega)$ and $\tilde{u} = h \circ u$, where h is increasing and piecewise continuously differentiable on $[0, \infty)$. Then \tilde{u} formally satisfies the equation

$$\mathcal{L}\tilde{u} = \nabla^{\text{tr}} A \nabla (h \circ u) = \nabla^{\text{tr}} A h'(u) \nabla u = h'(u) \mathcal{L}u + h''(u) (\nabla u)^{\text{tr}} A \nabla u,$$

and if u is a positive subsolution of (3.1) in Ω , we have

$$\begin{aligned} (3.3) \quad - \int (\nabla w)^{\text{tr}} A \nabla \tilde{u} &= \int w \mathcal{L}\tilde{u} = \int w h'(u) \mathcal{L}u + \int w h''(u) \|\nabla_A u\|^2 \\ &\geq \int w h'(u) \phi + \int w h''(u) \|\nabla_A u\|^2, \end{aligned}$$

provided $wh'(u)$ is nonnegative and in the space $(W_A^{1,2})_0(\Omega)$, which will be the case if in addition h' is bounded.

Note: We start with a weaker version of reverse Sobolev inequality, which we will apply with $u \equiv w + \|\phi\|_{X(B(0,r))}$ where w is a nonnegative subsolution to $\mathcal{L}w = \phi$ in $B(0, r)$.

LEMMA 33. Assume that u is a weak subsolution to $\mathcal{L}u = \phi$ in $B(0, r)$ and that

$$\inf_{x \in B(0,r)} u(x) > \|\phi\|_{X(B(0,r))} \in (0, \infty).$$

Let $h(t)$ be a piecewise continuously differentiable function that satisfies the following conditions when $t \geq \inf_{B(0,r)} u$:

- (I) $\Lambda(t) \equiv \left(\frac{1}{2}h(t)^2\right)''$ is positive;
- (II) $\Lambda(t) = h(t)h''(t) + h'(t)^2 \approx h'(t)^2$, so that in particular, we can assume $\Lambda(t) \geq C_1 h'(t)^2$ where $C_1 \leq 1$ is a constant;
- (III) The derivative $h'(t)$ satisfies the inequality $0 < h'(t) \leq C_2 \frac{h(t)}{t}$, where $C_2 \geq 1$ is a constant;

Then the following reverse Sobolev inequality holds for any $\psi \in C_0^{0,1}(B(0, r))$:

$$(3.4) \quad \int_{B(0,r)} \psi^2 \|\nabla_A [h(u)]\|^2 dx \leq \frac{21C_2^2}{C_1^2} \int_{B(0,r)} [h(u)]^2 (|\nabla_A \psi|^2 + \psi^2).$$

PROOF. Let us first prove the lemma with an apriori assumption that $h'(t)$ is bounded. This assumption can be dropped by the following limiting argument. Using standard truncations as in [SaWh4], we define for $N > E$,

$$h_N(t) \equiv \begin{cases} h(t) & \text{if } E \leq t \leq N \\ h(N) + h'(N)(t - N) & \text{if } t \geq N \end{cases}.$$

Observing that the function h_N still satisfies the admissible conditions (I), (II), (III) in the lemma with the same constants C_1 and C_2 , we can obtain a reverse Sobolev inequality similar to (3.4) if we substitute h by h_N . Now the monotone converge theorem applies to obtain (3.4).

Let $\psi \in C_0^{0,1}(B(0, r))$ and take $w = \psi^2 h(u)$. By the assumption that $h'(u)$ is positive and bounded, we have that $wh'(u) = \psi^2 h(u)h'(u)$ is nonnegative and in the space $(W_A^{1,2})_0(B(0, r))$. As a result, from the integral inequality (3.3) we obtain

$$(3.5) \quad \int \langle \nabla_A h(u), \nabla_A \psi^2 h(u) \rangle + \int \psi^2 h(u) h''(u) \|\nabla_A u\|^2 \leq - \int wh'(u) \phi.$$

The left side of (3.5) equals

$$(3.6) \quad \begin{aligned} \int \psi^2 h'(u)^2 \langle \nabla_A u, \nabla_A u \rangle &+ 2 \int \psi h(u) h'(u) \langle \nabla_A u, \nabla_A \psi \rangle + \int \psi^2 h(u) h''(u) \|\nabla_A u\|^2 \\ &= \int \psi^2 \Lambda(u) \|\nabla_A u\|^2 + 2 \int \langle \psi \nabla_A h(u), h(u) \nabla_A \psi \rangle, \end{aligned}$$

where $\Lambda(t) = h'(t)^2 + h(t)h''(t) = \left[\frac{1}{2}h(t)^2\right]''$. Combining (3.5) and (3.6) we obtain

$$\int \psi^2 \Lambda(u) \|\nabla_A u\|^2 + 2 \int \langle \psi \nabla_A h(u), h(u) \nabla_A \psi \rangle \leq - \int \psi^2 h(u) h'(u) \phi$$

For $0 < \varepsilon < 1$, we can estimate the last term on the left side above by

$$\begin{aligned} 2 \left| \int \langle \psi \nabla_A h(u), h(u) \nabla_A \psi \rangle \right| &\leq \varepsilon \int \langle \psi \nabla_A h(u), \psi \nabla_A h(u) \rangle + \varepsilon^{-1} \int \langle h(u) \nabla_A \psi, h(u) \nabla_A \psi \rangle \\ &= \varepsilon \int \psi^2 h'(u)^2 \|\nabla_A u\|^2 + \varepsilon^{-1} \int h(u)^2 \|\nabla_A \psi\|^2. \end{aligned}$$

Since $\Lambda(t)$ is positive, we have

$$(3.7) \quad \int \psi^2 \left\{ |\Lambda(u)| - \varepsilon h'(u)^2 \right\} \|\nabla_A u\|^2 \leq \varepsilon^{-1} \int h(u)^2 \|\nabla_A \psi\|^2 + \int |\psi^2 h(u) h'(u) \phi|$$

We can find an upper bound of the right hand by

$$(3.8) \quad \begin{aligned} \int |\psi^2 h(u) h'(u) \phi| &\leq C_2 \int_{B(0,r)} \psi^2 h(u)^2 \frac{|\phi|}{u} \leq \frac{C_2}{\inf_{B(0,r)} u} \int_{B(0,r)} |\psi^2 h(u)^2 \phi| \\ &\leq C_2 \int_{B(0,r)} \left\| \nabla_A (\psi^2 h(u)^2) \right\| \leq C_2 \int \left\{ |\nabla_A \psi|^2 h(u)^2 + \psi^2 2h(u) |\nabla_A h(u)| \right\} \\ &\leq C_2 \int (|\nabla_A \psi|^2 + \psi^2) h(u)^2 + C_2 \int \psi^2 \left(\frac{1}{\varepsilon_1} h(u)^2 + \varepsilon_1 |\nabla_A h(u)|^2 \right). \end{aligned}$$

Combining this with (3.7) and remembering that we are actually supposing $h = h_M$ so that all integrals are finite, we obtain

$$(3.9) \quad \int \psi^2 \left\{ |\Lambda(u)| - (\varepsilon + C_2 \varepsilon_1) h'(u)^2 \right\} |\nabla_A u|^2 \leq \left(\frac{1}{\varepsilon} + C_2 + \frac{C_2}{\varepsilon_1} \right) \int h(u)^2 (|\nabla_A \psi|^2 + \psi^2).$$

According to condition (II) for h , we obtain that $|\Lambda(u)| - (\varepsilon + C_2 \varepsilon_1) h'(u)^2 \geq (C_1 - \varepsilon - C_2 \varepsilon_1) |h'|^2$. As a result, we have

$$\int_{B(0,r)} \psi^2 \|\nabla_A [h(u)]\|^2 dx \leq C(\varepsilon, \varepsilon_1) \int_{B(0,r)} [h(u)]^2 (|\nabla_A \psi|^2 + \psi^2).$$

Here the constant C is given by

$$C(\varepsilon, \varepsilon_1) = \frac{\varepsilon^{-1} + C_2 + C_2 \varepsilon_1^{-1}}{C_1 - \varepsilon - C_2 \varepsilon_1}$$

Finally we can take $\varepsilon = C_1/3$, $\varepsilon_1 = C_1/3C_2$ and finish the proof. ■

Now we consider the specific family of examples that arise in our proof. Although there will be technical difficulties to be overcome, we wish to apply inequality (3.4) with

$$h(t) = \Gamma_m^{(n)}(t) \equiv \Gamma_m \circ \Gamma_m \circ \dots \Gamma_m(t),$$

where the function $\Gamma_m(t) \equiv \sqrt{\Phi_m(t^2)}$ for $m > 1$. When $t > e^{2^{m-1}}$, we have the explicit formula

$$\Gamma_m(t) \equiv \sqrt{\Phi_m(t^2)} = e^{\frac{1}{2}((2 \ln t)^{\frac{1}{m}} + 1)^m} > t.$$

A basic induction gives the formula $h(t) = e^{\frac{1}{2}((2 \ln t)^{\frac{1}{m}} + n)^m}$ for $t > e^{2^{m-1}}$. Therefore we have

$$\begin{aligned} h'(t) &= h(t) \left\{ \frac{m}{2} \left((2 \ln t)^{\frac{1}{m}} + n \right)^{m-1} \frac{1}{m} (2 \ln t)^{\frac{1}{m}-1} \frac{2}{t} \right\} \\ &= h(t) \left\{ \left((2 \ln t)^{\frac{1}{m}} + n \right)^{m-1} (2 \ln t)^{-\frac{m-1}{m}} \frac{1}{t} \right\} = h(t) \left\{ \left(1 + n (2 \ln t)^{-\frac{1}{m}} \right)^{m-1} \frac{1}{t} \right\}, \end{aligned}$$

Thus we can choose the constant $C_2 = (1 + n/2)^{m-1}$. In addition

$$\begin{aligned} h''(t) &= \frac{d}{dt} \left\{ h(t) \left[\left(1 + n(2 \ln t)^{-\frac{1}{m}} \right)^{m-1} \frac{1}{t} \right] \right\} \\ &= h(t) \left[\left(1 + n(2 \ln t)^{-\frac{1}{m}} \right)^{m-1} \frac{1}{t} \right]^2 \\ &\quad + h(t) \left\{ (m-1) \left(1 + n(2 \ln t)^{-\frac{1}{m}} \right)^{m-2} \left(-\frac{n}{m} \right) (2 \ln t)^{-\frac{1}{m}-1} \frac{2}{t^2} - \left(1 + n(2 \ln t)^{-\frac{1}{m}} \right)^{m-1} \frac{1}{t^2} \right\} \end{aligned}$$

Using the notation $\eta = (2 \ln t)^{-1/m} < 1/2$, we can rewrite

$$\begin{aligned} h''(t) &= \frac{h(t)}{t^2} \left\{ (1 + n\eta)^{2m-2} - \frac{2n(m-1)}{m} \eta^{m+1} (1 + n\eta)^{m-2} - (1 + n\eta)^{m-1} \right\} \\ &= \frac{h(t)}{t^2} (1 + n\eta)^{m-2} \left\{ (1 + n\eta)^m - \frac{2n(m-1)}{m} \eta^{m+1} - (1 + n\eta) \right\} \end{aligned}$$

Since $(1 + n\eta)^m > 1 + nm\eta$ when $m > 1$, we have

$$h''(t) > \frac{h(t)}{t^2} (1 + n\eta)^{m-2} \cdot (m-1)n\eta \left\{ 1 - \frac{2\eta^m}{m} \right\} > 0.$$

This implies $\Lambda(t) = h(t)h''(t) + h'^2 \geq h'^2$ thus we can choose $C_1 \equiv 1$.

2. Sub solutions of the form $\Gamma^{(n)}(u^\beta)$, $-\frac{1}{2} < \beta < 0$, with solution $u > M$

We begin with a variant of Lemma 33 for weak solutions, and where h can now be decreasing.

LEMMA 34. Assume that u is a weak solution to $\mathcal{L}u = \phi$ in $B(0, r)$ so that

$$\inf_{x \in B(0, r)} u(x) > \|\phi\|_{X(B(0, r))} \in (0, \infty).$$

Let $h(t)$ be a piecewise continuously differentiable function that satisfies the following conditions when $t \geq \inf_{B(0, r)} u$:

- (I) $\Lambda(t) \equiv \left(\frac{1}{2} h(t)^2 \right)''$ is positive;
- (II) $\Lambda(t) = h(t)h''(t) + h'(t)^2 \approx h'(t)^2$, so that in particular, we can assume $|\Lambda(t)| \geq C_1 h'(t)^2$ where $C_1 \leq 1$ is a constant;
- (III') The derivative $h'(t)$ satisfies the inequality $|h'(t)| \leq C_2 \frac{h(t)}{t}$, where $C_2 \geq 1$ is a constant;

Then the following reverse Sobolev inequality holds for any $\psi \in C_0^{0,1}(B(0, r))$:

$$(3.10) \quad \int_{B(0, r)} \psi^2 \|\nabla_A [h(u)]\|^2 dx \leq \frac{21C_2^2}{C_1^2} \int_{B(0, r)} [h(u)]^2 (|\nabla_A \psi|^2 + \psi^2).$$

PROOF. The proof is virtually identical to that of Lemma 33 upon noting that (3.5) now holds with equality. ■

Now consider $h_\beta(t) \equiv \sqrt{\Phi^{(n)}(t^{2\beta})} \equiv \Gamma_m^{(n)}(t^\beta)$. We wish to show that this h satisfies the conditions of Lemma 34 above. So let $\beta \in (-\frac{1}{2}, 0)$. We have

$$\Gamma_m(t^\beta) = \begin{cases} \tau t^\beta & t^\beta \leq e^{2^{m-1}}; \\ e^{\frac{1}{2}((2\beta \ln t)^{1/m} + 1)^m} & t^\beta \geq e^{2^{m-1}}; \end{cases}$$

where $\tau = e^{\frac{3^m - 2^m}{2}}$ as before. For the n^{th} iteration this gives

$$\Gamma_m^{(n)}(t^\beta) = \begin{cases} \tau^n t^\beta & t^\beta \leq \tau^{-(n-1)} e^{2^{m-1}}; \\ e^{\frac{1}{2}((2(n-k) \ln \tau + 2\beta \ln t)^{1/m} + k)^m} & \tau^{-(n-k)} e^{2^{m-1}} \leq t^\beta \leq \tau^{-(n-k-1)} e^{2^{m-1}}, \quad k = 1, 2, \dots, n-1; \\ e^{\frac{1}{2}((2\beta \ln t)^{1/m} + n)^m} & t^\beta \geq e^{2^{m-1}} \end{cases}.$$

Recall $h_\beta(t) = \Gamma_m^{(n)}(t^\beta)$ and $\Lambda_\beta(t) = (h_\beta(t)^2)''/2$. Then for $t^\beta \leq \tau^{-(n-1)} e^{2^{m-1}}$ it is easy to calculate

$$\Lambda_\beta(t) = \tau^{2n} 2\beta(2\beta - 1)t^{2\beta-2} = \frac{2(2\beta - 1)}{\beta} (h'_\beta(t))^2 > 0,$$

and the coefficient of $(h'_\beta(t))^2$ is strictly positive for the range of β chosen. For the other values of t we have

$$|h'_\beta(t)| = \frac{h_\beta(t)}{t} \left(1 + \frac{k}{(2 \ln(\tau^{n-k} t^\beta))^{\frac{1}{m}}} \right)^{m-1} \cdot |\beta| \leq |\beta| \frac{h_\beta(t)}{t} (k+1)^{m-1},$$

and

$$\begin{aligned} \Lambda_\beta(t) &= \frac{h_\beta(t)^2}{t^2} \left(1 + \frac{k}{(2 \ln(\tau^{n-k} t^\beta))^{\frac{1}{m}}} \right)^{m-2} \cdot \beta^2 \\ &\quad \cdot \left(2 \left(1 + \frac{k}{(2 \ln(\tau^{n-k} t^\beta))^{\frac{1}{m}}} \right)^m - \frac{1}{\beta} \left(1 + \frac{k}{(2 \ln(\tau^{n-k} t^\beta))^{\frac{1}{m}}} \right) - \frac{2(m-1)}{m} \frac{k}{(2 \ln(\tau^{n-k} t^\beta))^{\frac{m+1}{m}}} \right). \end{aligned}$$

Thus since $\tau^{n-k} t^\beta \geq e^{2^{m-1}}$ we get

$$\frac{2(m-1)}{m} \frac{k}{(2 \ln(\tau^{n-k} t^\beta))^{\frac{m+1}{m}}} \leq -\frac{1}{\beta} \left(1 + \frac{k}{(2 \ln(\tau^{n-k} t^\beta))^{\frac{1}{m}}} \right),$$

which altogether shows that

$$\Lambda_\beta(t) \approx |h'_\beta(t)|^2 \text{ for all } t.$$

Moreover, we also have

$$|h'_\beta(t)| \leq |\beta| \frac{h_\beta(t)}{t} (k+1)^{m-1} \leq C n^{m-1} \frac{h_\beta(t)}{t}.$$

Thus h_β satisfies the hypotheses of Lemma 34 and so we conclude that

$$(3.11) \quad \int_{B(0,r)} \psi^2 \|\nabla_A [h(u)]\|^2 dx \leq \frac{C_2^2}{C_1^2} \int_{B(0,r)} [h(u)]^2 (|\nabla_A \psi|^2 + \psi^2),$$

where $C_2 = C n^{m-1}$ and C_1 is as in (II) above.

3. Sub and super solutions of the form $\Psi^{(n)}(\Psi^{(-N)}(u))$ with solution $u < \frac{1}{M}$

The major difficulty encountered in establishing a Cacciopoli inequality for small solutions is that the function $h(t) = \sqrt{\Psi^{(-1)}(t)}$ no longer satisfies the equivalence $\Lambda_h(t) \approx |h'(t)|^2$ in condition (II) of Lemmas 33 and 34, in fact $\lim_{t \rightarrow 0} \frac{\Lambda(t)}{h'(t)^2} = 0$ as follows easily from (3.15) below. Previously, we used a nonlinear function $h(t) = \sqrt{\Phi(t^2)}$ where the strong convexity of t^2 ensured the equivalence $\Lambda_h(t) \approx |h'(t)|^2$.

3.1. A preliminary Cacciopoli inequality. Recall that for any functions h and u the composition $\tilde{u} \equiv h(u)$ formally satisfies the equation

$$\mathcal{L}\tilde{u} = \nabla^{\text{tr}} A \nabla (h \circ u) = \nabla^{\text{tr}} A h'(u) \nabla u = h'(u) \mathcal{L}u + h''(u) (\nabla u)^{\text{tr}} A \nabla u,$$

and if u is a positive supersolution of (3.1) in Ω , i.e. $\mathcal{L}u = \phi$, then we have

$$\begin{aligned} - \int (\nabla w)^{\text{tr}} A \nabla \tilde{u} &= \int w \mathcal{L}\tilde{u} = \int w h'(u) \mathcal{L}u + \int w h''(u) \|\nabla_A u\|^2 \\ &\leq \int w h'(u) \phi + \int w h''(u) \|\nabla_A u\|^2, \end{aligned}$$

provided $wh'(u)$ is nonnegative and in the space $(W_A^{1,2})_0(\Omega)$, which will be the case if in addition h' is bounded. If we substitute $w = \psi^2 h(u)$ we get

$$\begin{aligned} & - \int \psi^2 \|\nabla_A h(u)\|^2 - 2 \int \langle h(u) \nabla \psi, \psi \nabla h(u) \rangle_A \\ &= - \int (\nabla [\psi^2 h(u)])^{\text{tr}} A \nabla h(u) \\ &\leq \int \psi^2 h(u) h'(u) \phi + \int \psi^2 h(u) h''(u) \|\nabla_A u\|^2, \end{aligned}$$

hence

$$\begin{aligned} (3.12) \quad - \int \psi^2 \Lambda(u) \|\nabla_A u\|^2 &= - \int \psi^2 \{h(u) h''(u) + h'(u)^2\} \|\nabla_A u\|^2 \\ &\leq 2 \int \langle h(u) \nabla_A \psi, \psi \nabla_A h(u) \rangle + \int \psi^2 h(u) h'(u) \phi. \end{aligned}$$

REMARK 35. Suppose that $h(t)$ is increasing with $h(0) = 0$. If $h(t)^2$ is concave, then so is $h(t) > 0$, and hence $h'(t) \leq \frac{h(t)}{t}$. Thus condition (III) below is redundant, but is included for emphasis.

LEMMA 36. Suppose $\psi \in \text{Lip}_0(B(0, r))$. Let $u < \frac{1}{M}$ be a weak supersolution to $Lu = \phi$ with ϕ admissible and let $h(t)$ be a piecewise continuously differentiable function that satisfies the following two conditions when $\frac{1}{M} > t \geq \inf_{B(0, r)} u$:

(I) $\Lambda(t) \equiv (\frac{1}{2}h(t)^2)''$ is negative;

(III) The derivative $h'(t)$ satisfies the inequality $h'(t) \leq \frac{h(t)}{t}$.

Then the following Cacciopoli inequality holds:

$$(3.13) \quad \int \psi^2 \|\nabla_A k(u)\|^2 \leq C \int \left(\psi^2 + \|\nabla_A \psi\|^2 \right) \frac{h'(u)^2}{|\Lambda(u)|} h(u)^2,$$

where

$$k(t) = \int_0^t \sqrt{|\Lambda(s)|} ds, \quad \Lambda(t) = \frac{1}{2} (h^2(t))''.$$

PROOF. The Cacciopoli inequality when u is a supersolution is (3.12):

$$- \int \psi^2 \Lambda(v) \|\nabla_A u\|^2 - 2 \int \langle \psi \nabla_A h(u), h(u) \nabla_A \psi \rangle \leq \int \psi^2 h(u) h'(u) \phi.$$

For $0 < \varepsilon < 1$, we can estimate the last term on the left side above by

$$\begin{aligned}
2 \left| \int \langle \psi \nabla_A h(u), h(u) \nabla_A \psi \rangle \right| &= 2 \left| \int \langle \psi h'(u) \nabla_A u, h(u) \nabla_A \psi \rangle \right| \\
&= 2 \left| \int \left\langle \psi \sqrt{|\Lambda(u)|} \nabla_A u, \frac{h'(u)}{\sqrt{|\Lambda(u)|}} h(u) \nabla_A \psi \right\rangle \right| \\
&\leq \varepsilon \int \langle \psi \sqrt{|\Lambda(u)|} \nabla_A u, \psi \sqrt{|\Lambda(u)|} \nabla_A u \rangle \\
&\quad + \varepsilon^{-1} \int \left\langle \frac{h'(u)}{\sqrt{|\Lambda(u)|}} h(u) \nabla_A \psi, \frac{h'(u)}{\sqrt{|\Lambda(u)|}} h(u) \nabla_A \psi \right\rangle \\
&= \varepsilon \int \psi^2 |\Lambda(u)| \|\nabla_A u\|^2 + \varepsilon^{-1} \int \frac{|h'(u)|^2}{|\Lambda(u)|} h(u)^2 \|\nabla_A \psi\|^2.
\end{aligned}$$

Since $\Lambda(t)$ is negative, we have

$$(1 - \varepsilon) \int \psi^2 |\Lambda(u)| \|\nabla_A u\|^2 \leq \varepsilon^{-1} \int \frac{|h'(u)|^2}{|\Lambda(u)|} h(u)^2 \|\nabla_A \psi\|^2 + \int |\psi^2 h(u) h'(u) \phi|.$$

In the case $\phi = 0$ we have

$$\int \psi^2 |\Lambda(u)| \|\nabla_A u\|^2 \leq C_\varepsilon \int \frac{|h'(u)|^2 h(u)^2}{|\Lambda(u)|} \|\nabla_A \psi\|^2,$$

which gives (3.13). If $\phi \neq 0$ and $u \geq \|\phi\|_{X(B(0,r))}$, then using (III) we have the bound

$$\begin{aligned}
\int |\psi^2 h(u) h'(u) \phi| &\leq C_2 \int_{B(0,r)} \psi^2 h(u)^2 \frac{|\phi|}{u} \leq \frac{C_2}{\inf_{B(0,r)} u} \int_{B(0,r)} |\psi^2 h(u)^2 \phi| \\
&\leq C_2 \frac{\|\phi\|_{X(B(0,r))}}{\inf_{B(0,r)} u} \int_{B(0,r)} \|\nabla_A (\psi^2 h(u)^2)\| \leq C_2 \int \left\{ |\nabla_A \psi^2| h(u)^2 + \psi^2 2h(u) h'(u) |\nabla_A u| \right\} \\
&\leq C_2 \int (|\nabla_A \psi|^2 + \psi^2) h(u)^2 + 2C_2 \int \psi^2 \frac{h(u) h'(u)}{\sqrt{|\Lambda(t)|}} |\nabla_A k(u)| \\
&\leq C_2 \int (|\nabla_A \psi|^2 + \psi^2) h(u)^2 + C_2 \int \psi^2 \left(\frac{1}{\varepsilon_1} \frac{h(u)^2 (h'(u))^2}{|\Lambda(t)|} + \varepsilon_1 |\nabla_A k(u)|^2 \right).
\end{aligned}$$

Now we can absorb the term $C_2 \varepsilon_1 \int \psi^2 |\nabla_A k(u)|^2 = C_2 \varepsilon_1 \int \psi^2 |\Lambda(u)| |\nabla_A u|^2$, and use that $\frac{|h'(u)|^2}{|\Lambda(u)|} \geq c > 0$. ■

3.2. Iterates of concave functions. In order to obtain a Cacciopoli inequality suitable for iterating with an inhomogeneous Orlicz-Sobolev bump inequality, we first establish some estimates on the iterated function $\Psi^{(-N)}$. Set

$$\begin{aligned}
(3.14) \quad H(t) &\equiv \Psi^{(-1)}(t) = e^{-\left((\ln \frac{4}{t})^{\frac{1}{m}} - 1\right)^m}, \\
H_N(t) &\equiv \Psi^{(-N)}(t) \text{ for } N \geq 1 \text{ and } H_0(t) \equiv t, \\
h_N(t) &\equiv \sqrt{H_N(t)} \text{ for } N \geq 1, \\
\Lambda_N(t) &\equiv \frac{1}{2} H_N''(t) = h_N(t) h_N''(t) + |h_N'(t)|^2 \text{ for } N \geq 1.
\end{aligned}$$

Then we have

$$\frac{\Lambda_N(t)}{|h'_N(t)|^2} = \frac{\frac{1}{2}H''_N(t)}{\left|\frac{1}{2}\frac{H'_N(t)}{\sqrt{H_N(t)}}\right|^2} = 2\frac{H_N(t)H''_N(t)}{|H'_N(t)|^2}.$$

We next compute H' and H'' , and for this it is convenient to write

$$H(t) = e^{-\left(\ln \frac{A}{t}\right)\left(1 - \left(\ln \frac{A}{t}\right)^{-\frac{1}{m}}\right)^m} = t^{\left(1 - \left(\ln \frac{A}{t}\right)^{-\frac{1}{m}}\right)^m},$$

and to introduce

$$\begin{aligned}\Omega(t) &\equiv \left(1 - \left(\ln \frac{1}{t}\right)^{-\frac{1}{m}}\right)^{m-1}, \\ \Omega'(t) &\equiv -(m-1)\left(1 - \left(\ln \frac{1}{t}\right)^{-\frac{1}{m}}\right)^{m-2} \frac{1}{m} \left(\ln \frac{1}{t}\right)^{-\frac{1}{m}-1} \frac{1}{t} \\ &= -\frac{m-1}{m} \left(1 - \left(\ln \frac{1}{t}\right)^{-\frac{1}{m}}\right)^{m-2} \frac{1}{t \left(\ln \frac{1}{t}\right)^{\frac{m+1}{m}}} \\ &= -\frac{m-1}{m} \frac{\Omega(t)^{\frac{m-2}{m-1}}}{t \left(\ln \frac{1}{t}\right)^{\frac{m+1}{m}}}.\end{aligned}$$

Then we have

$$\begin{aligned}H'(t) &= H(t) \frac{d}{dt} \left\{ - \left(\left(\ln \frac{1}{t} \right)^{\frac{1}{m}} - 1 \right)^m \right\} \\ &= \frac{H(t)}{t} \left(1 - \frac{1}{\left(\ln \frac{1}{t} \right)^{\frac{1}{m}}} \right)^{m-1} = \frac{H(t)\Omega(t)}{t},\end{aligned}$$

and

$$\begin{aligned}(3.15) \quad H''(t) &= \frac{H(t)\Omega(t)^2}{t^2} - \frac{H(t)\Omega(t)}{t^2} + \frac{H(t)\Omega'(t)}{t} \\ &= -\frac{H(t)\Omega(t)}{t^2} (1 - \Omega(t)) - \frac{H(t)}{t} \frac{m-1}{m} \frac{\Omega(t)^{\frac{m-2}{m-1}}}{t \left(\ln \frac{1}{t}\right)^{\frac{m+1}{m}}} \\ &= -\frac{H(t)\Omega(t)}{t^2} \left\{ 1 - \left(1 - \left(\ln \frac{1}{t} \right)^{-\frac{1}{m}} \right)^{m-1} + \frac{m-1}{m} \frac{\Omega(t)^{-\frac{1}{m-1}}}{\left(\ln \frac{1}{t} \right)^{\frac{m+1}{m}}} \right\} \\ &\equiv -\frac{H(t)\Omega(t)}{t^2} \Gamma(t),\end{aligned}$$

where

$$\begin{aligned}\Gamma(t) &= 1 - \left(1 - \left(\ln \frac{1}{t}\right)^{-\frac{1}{m}}\right)^{m-1} + \frac{m-1}{m} \frac{\Omega(t)^{-\frac{1}{m-1}}}{\left(\ln \frac{1}{t}\right)^{\frac{m+1}{m}}} \\ &\geq 1 - \left(1 - \left(\ln \frac{1}{t}\right)^{-\frac{1}{m}}\right)^{m-1} \geq \frac{m - \frac{3}{2}}{\left(\ln \frac{1}{t}\right)^{\frac{1}{m}}},\end{aligned}$$

for $0 < t < \frac{1}{M}$. Now we use the composition formulae

$$\begin{aligned}(f \circ g)' &= (f' \circ g) g', \\ (f \circ g)'' &= (f'' \circ g) |g'|^2 + (f' \circ g) g'',\end{aligned}$$

to compute $\frac{\Lambda_N(t)}{|h'_N(t)|^2}$ for $N \geq 1$. Indeed, using $H'(t) = \frac{H(t)}{t}\Omega(t)$ and $H''(t) \approx -\frac{H(t)}{t^2}\Gamma(t)$, we have

$$\begin{aligned}H_N(t) &= H \circ H_{N-1}(t), \\ (H_N)'(t) &= H'(H_{N-1}(t)) H'_{N-1}(t) = \frac{H_N(t)}{H_{N-1}(t)} H'_{N-1}(t) \Omega(H_{N-1}(t)), \\ (H_N)''(t) &= H''(H_{N-1}(t)) |H'_{N-1}(t)|^2 + H'(H_{N-1}(t)) H''_{N-1}(t) \\ &= -\frac{H((H_{N-1})(t))}{H_{N-1}(t)^2} \Gamma(H_{N-1}(t)) |H'_{N-1}(t)|^2 + \frac{H(H_{N-1}(t))}{H_{N-1}(t)} \Omega(H_{N-1}(t)) H''_{N-1}(t) \\ &= -\frac{H_N(t)}{H_{N-1}^2(t)} \Gamma(H_{N-1}(t)) |H'_{N-1}(t)|^2 + \frac{H_N(t)}{H_{N-1}(t)} H''_{N-1}(t) \Omega(H_{N-1}(t)),\end{aligned}$$

Hence we have

$$\begin{aligned}H_N(t) (H_N)''(t) &= -\frac{H_N(t)^2}{H_{N-1}(t)^2} |H'_{N-1}(t)|^2 \Gamma(H_{N-1}(t)) + \frac{H_N(t)^2}{H_{N-1}(t)} H''_{N-1}(t) \\ &= -|(H_N)'(t)|^2 \frac{\Gamma(H_{N-1}(t))}{\Omega(H_{N-1}(t))^2} + \frac{H_N(t)^2}{H_{N-1}(t)^2} H_{N-1}(t) H''_{N-1}(t) \\ &= -|(H_N)'(t)|^2 \frac{\Gamma(H_{N-1}(t))}{\Omega(H_{N-1}(t))^2} + \frac{|H'_N(t)|^2}{\Omega(H_{N-1}(t))^2} \frac{H_{N-1}(t) H''_{N-1}(t)}{|H'_{N-1}(t)|^2},\end{aligned}$$

which gives

$$\frac{H_N(t) |(H_N)''(t)|}{|(H_N)'(t)|^2} = \frac{1}{\Omega(H_{N-1}(t))^2} \left(\Gamma(H_{N-1}(t)) + \frac{H_{N-1}(t) |H''_{N-1}(t)|}{|H'_{N-1}(t)|^2} \right), \quad \text{for } N \geq 1.$$

Now $\Omega(t) > 1$ and $\frac{H_{N-1}(t) |H''_{N-1}(t)|}{|H'_{N-1}(t)|^2} > 0$ so we trivially have the lower bound

$$\frac{H_N(t) |(H_N)''(t)|}{|(H_N)'(t)|^2} \geq \Gamma(H_{N-1}(t)) \geq \frac{m - \frac{3}{2}}{\left(\ln \frac{1}{H_{N-1}(t)}\right)^{\frac{1}{m}}}, \quad 0 < t < \frac{1}{M}.$$

We summarize these calculations in the following lemma.

LEMMA 37. For $N \geq 1$ we have

$$2 \frac{|h'_N(t)|^2}{|\Lambda_N(t)|} = \frac{|H'_N(t)|^2}{H_N(t) H''_N(t)} \leq \frac{1}{m - \frac{3}{2}} \left(\ln \frac{1}{H_{N-1}(t)} \right)^{\frac{1}{m}}, \quad 0 < t < \frac{1}{M}.$$

Now we express the right hand side above as a composition with $H_N(t)$. Let

$$(3.16) \quad \Theta(t) = \left(\ln \frac{1}{\Psi(t)} \right)^{\frac{1}{m}} t, \quad 0 < t < \frac{1}{M}.$$

Then since $\Psi(H_N(t)) = H_{N-1}(t)$ we have

$$\left(\ln \frac{1}{H_{N-1}(t)} \right)^{\frac{1}{m}} H_N(t) = \left(\ln \frac{1}{\Psi(H_N(t))} \right)^{\frac{1}{m}} H_N(t) = \Theta(H_N(t)).$$

We now claim that Θ is concave. Indeed, from $\Psi(t) = Ae^{-(\ln \frac{1}{t})^{\frac{1}{m}+1}}$ we have

$$\left(\ln \frac{1}{\Psi(t)} \right) = \ln \frac{1}{A} + \left(\ln \frac{1}{t} \right)^{\frac{1}{m}+1},$$

and so

$$\begin{aligned} \frac{d}{dt} \left(\ln \frac{1}{\Psi(t)} \right) &= \frac{d}{dt} \left(\left(\ln \frac{1}{t} \right)^{\frac{1}{m}+1} \right) = m \left(\left(\ln \frac{1}{t} \right)^{\frac{1}{m}} + 1 \right)^{m-1} \frac{1}{m} \left(\ln \frac{1}{t} \right)^{\frac{1}{m}-1} \left(-\frac{1}{t} \right) \\ &= -\frac{1}{t} \left(1 + \left(\ln \frac{1}{t} \right)^{-\frac{1}{m}} \right)^{m-1}, \end{aligned}$$

and then

$$\Theta'(t) = \frac{d}{dt} \left[t \left(\ln \frac{1}{\Psi(t)} \right)^{\frac{1}{m}} \right] = \left(\ln \frac{1}{\Psi(t)} \right)^{\frac{1}{m}} - \frac{1}{m} \left(\ln \frac{1}{\Psi(t)} \right)^{\frac{1}{m}-1} \left(1 + \left(\ln \frac{1}{t} \right)^{-\frac{1}{m}} \right)^{m-1}.$$

Now both $\ln \frac{1}{\Psi(t)}$ and $\ln \frac{1}{t}$ are decreasing, and hence $\Theta'(t)$ is decreasing, and so Θ is concave.

We collect all of these observations in the next lemma.

LEMMA 38. With notation as above, we have

$$\begin{aligned} 2 \frac{|h'_N(t)|^2}{|\Lambda_N(t)|} h_N(t)^2 &= \frac{|H'_N(t)|^2}{H_N(t) H''_N(t)} H_N(t) \\ &\leq \frac{1}{m - \frac{3}{2}} \left(\ln \frac{1}{H_{N-1}(t)} \right)^{\frac{1}{m}} H_N(t) \\ &= \Theta(H_N(t)) = \Theta(h_N(t)^2). \end{aligned}$$

3.3. A modified Cacciopoli inequality. Now we are prepared to extend the preliminary Lemma 36 to a Cacciopoli inequality for sub and super solutions of the form $\Psi^{(n)}(\Psi^{(-N)}(u))$ with a positive solution $u < \frac{1}{M}$.

LEMMA 39. Fix $N \geq 1$. Let $u < \frac{1}{M}$ be a weak supersolution to $Lu = \phi$ with ϕ admissible. Let $h_N(t) \equiv \sqrt{\Psi^{(-N)}(t)}$ and $\Theta(t)$ be as above. Then the following Cacciopoli inequality holds:

$$\|\nabla_A k_N(u)\|_{L^2(B_{n+1})}^2 \leq C_n(r)^2 \left\| \frac{h'_N(v)h_N(v)}{\sqrt{|\Lambda_N(v)|}} \right\|_{L^2(B_n)}^2 \leq \frac{C_n(r)^2}{m - \frac{3}{2}} \Theta \left(\|h_N(u)\|_{L^2(B_n)}^2 \right),$$

where

$$(3.17) \quad k_N(t) = \int_0^t \sqrt{|\Lambda_N(s)|} ds, \quad \Lambda_N(t) = \Psi^{(-N)}(t)''.$$

Similarly, if $u < \frac{1}{M}$ is a weak subsolution to $Lu = \phi$ with ϕ admissible, then

$$\|\nabla_A k_N(u)\|_{L^2(B_{n+1})}^2 \leq C_n(r)^2 \left\| \frac{h'(v)h(v)}{\sqrt{|\Lambda(v)|}} \right\|_{L^2(B_n)}^2 \leq \frac{C_n(r)^2}{m - \frac{3}{2}} \Theta \left(\|h_N(u)\|_{L^2(B_n)}^2 \right),$$

where now

$$k_N(t) = \int_0^t \sqrt{|\Lambda_N(s)|} ds, \quad \Lambda_N(t) = \Psi^{(N)}(t)''.$$

PROOF. From (3.13) we obtain

$$\|\nabla_A k_N(u)\|_{L^2(B_{n+1})}^2 \leq C_n(r)^2 \left\| \frac{h'_N(u)h_N(u)}{\sqrt{|\Lambda_N(u)|}} \right\|_{L^2(B_n)}^2 = C_n(r)^2 \int_{B_n} \frac{h'_N(u)^2}{|\Lambda_N(u)|} h_N(u)^2 d\mu.$$

Then from Lemma 38 we have

$$\int \frac{|h'_N(u)|^2}{|\Lambda_N(u)|} h_N(u)^2 d\mu \leq \frac{1}{m - \frac{3}{2}} \int \Theta(H_N(u)) d\mu.$$

Altogether we now have

$$\begin{aligned} \|\nabla_A k_N(u)\|_{L^2(B_{n+1})}^2 &\leq C_n(r)^2 \int_{B_n} \frac{h'_N(u)^2}{|\Lambda_N(u)|} h_N(u)^2 d\mu \\ &\leq \frac{C_n(r)^2}{m - \frac{3}{2}} \int \Theta(H_N(u)) d\mu = \frac{C_n(r)^2}{m - \frac{3}{2}} \Theta \circ \Theta^{(-1)} \left(\int \Theta(H_N(u)) d\mu \right) \\ &\leq \frac{C_n(r)^2}{m - \frac{3}{2}} \Theta \left(\int H_N(u) d\mu \right) = \frac{C_n(r)^2}{m - \frac{3}{2}} \Theta \left(\int h_N(u)^2 d\mu \right), \end{aligned}$$

where the final line follows by applying Jensen's inequality with the convex function $\Theta^{(-1)}$. This proves the first part of the lemma.

If we assume that u is a subsolution to $Lu = \phi$, and if we replace $\Psi^{(-1)}$ with Ψ , then the above arguments go through with $h_N(t) = \sqrt{\Psi^{(N)}(t)}$ and obvious modifications. Indeed, with $H(t) \equiv \Psi(t) = Ae^{-\left(\ln \frac{1}{t}\right)^{\frac{1}{m}+1}}$ and $H_N(t) \equiv \Psi^{(N)}(t)$ and

$$\begin{aligned} \widehat{\Omega}(t) &= \left(1 + \left(\ln \frac{1}{t} \right)^{-\frac{1}{m}} \right)^{m-1}, \\ \widehat{\Omega}'(t) &= \frac{m-1}{m} \frac{\widehat{\Omega}(t)^{\frac{m-2}{m-1}}}{t \left(\ln \frac{1}{t} \right)^{\frac{m+1}{m}}}, \end{aligned}$$

we have

$$\begin{aligned}
H'(t) &= \frac{H(t)}{t} \left(1 + \left(\ln \frac{1}{t} \right)^{-\frac{1}{m}} \right)^{m-1} = \frac{H(t) \widehat{\Omega}(t)}{t}, \\
H''(t) &= \frac{H(t) \widehat{\Omega}(t)^2}{t^2} - \frac{H(t) \widehat{\Omega}(t)}{t^2} + \frac{H(t) \widehat{\Omega}'(t)}{t} \\
&= \frac{H(t) \widehat{\Omega}(t)}{t^2} (\widehat{\Omega}(t) - 1) + \frac{H(t)}{t} \frac{m-1}{m} \frac{\widehat{\Omega}(t)^{\frac{m-2}{m-1}}}{t \left(\ln \frac{1}{t} \right)^{\frac{m+1}{m}}} \\
&= \frac{H(t) \widehat{\Omega}(t)}{t^2} \widehat{\Gamma}(t),
\end{aligned}$$

where

$$\begin{aligned}
\widehat{\Gamma}(t) &= \left(1 + \left(\ln \frac{1}{t} \right)^{-\frac{1}{m}} \right)^{m-1} - 1 + \frac{m-1}{m} \frac{\widehat{\Omega}(t)^{-\frac{1}{m-1}}}{\left(\ln \frac{1}{t} \right)^{\frac{m+1}{m}}} \\
&\geq 1 - \left(1 + \left(\ln \frac{1}{t} \right)^{-\frac{1}{m}} \right)^{m-1} \geq \frac{m-1}{\left(\ln \frac{1}{t} \right)^{\frac{1}{m}}}.
\end{aligned}$$

Thus H and H_N are convex and we compute that

$$\frac{H_N(t) (H_N)''(t)}{|(H_N)'(t)|^2} = \frac{1}{\widehat{\Omega}(H_{N-1}(t))^2} \left(\widehat{\Gamma}(H_{N-1}(t)) + \frac{H_{N-1}(t) H_{N-1}''(t)}{|H_{N-1}'(t)|^2} \right), \quad \text{for } N \geq 1,$$

and hence that

$$\frac{H_N(t) (H_N)''(t)}{|(H_N)'(t)|^2} \geq \frac{1}{\widehat{\Omega}(H_{N-1}(t))^2} \frac{m-1}{\left(\ln \frac{1}{H_{N-1}(t)} \right)^{\frac{1}{m}}} \geq \frac{m - \frac{3}{2}}{\left(\ln \frac{1}{H_{N-1}(t)} \right)^{\frac{1}{m}}}.$$

From this point on the arguments are essentially the same as for the case already considered, upon using that $\Lambda_N > 0$ and u is a subsolution. This completes the proof of the modified Cacciopoli inequality in Lemma 39. ■

CHAPTER 4

Local boundedness and maximum principle for weak sub solutions u

In this chapter, we use some of the Cacciopoli inequalities from the previous chapter to prove local boundedness of all weak subsolutions to $\mathcal{L}u = \phi$ with ϕ admissible under appropriate hypotheses including a Sobolev Orlicz bump inequality. In part 3 of the paper we will establish the corresponding geometric theorem.

1. Moser inequalities for sub solutions $u > M$

Here we assume that the inhomogeneous (Φ, φ) -Sobolev Orlicz bump inequality (1.5) holds. Let us start by considering $r > 0$ and the *standard* sequence of Lipschitz cutoff functions $\{\psi_j\}_{j=1}^\infty$ depending on r , along with the sets $B(0, r_j) \supset \text{supp } \psi_j$, so that $r_1 = r$, $r_\infty \equiv \lim_{j \rightarrow \infty} r_j = \frac{1}{2}$, $r_j - r_{j+1} = \frac{c}{j^2} r$ for a uniquely determined constant c , and $\|\nabla_A \psi_j\|_\infty \lesssim \frac{j^2}{r}$ with ∇_A as in (1.10) (see e.g. [SaWh4]). We apply Lemma 33 with $h(t) = \sqrt{\Phi_m^{(n)}(t^2)}$, $\psi = \psi_n$ and $\mu_{r_n} \equiv \mu_{0, r_n}$ as in Definition 6, to obtain

$$\int_{B(0, r_n)} \psi_n^2 \|\nabla_A [h(u)]\|^2 d\mu_{r_n} \leq 21 \left(1 + \frac{n}{2}\right)^{m-1} \int_{B(0, r_n)} [h(u)]^2 (|\nabla_A \psi_n|^2 + \psi_n^2) d\mu_{r_n}.$$

This implies

$$\begin{aligned} \|\nabla_A [\psi_n h(u)]\|_{L^2(\mu_{r_n})}^2 &\leq 2 \|\psi_n \nabla_A h(u)\|_{L^2(\mu_{r_n})}^2 + 2 \|\nabla_A \psi_n |h(u)|\|_{L^2(\mu_{r_n})}^2 \\ &\leq 42 \left(1 + \frac{n}{2}\right)^{m-1} \int_{B(0, r_n)} [h(u)]^2 (|\nabla_A \psi_n|^2 + \psi_n^2) d\mu_{r_n} + 2 \|\nabla_A \psi_n |h(u)|\|_{L^2(\mu_{r_n})}^2 \\ &\leq 86 \left(1 + \frac{n}{2}\right)^{m-1} \|\nabla_A \psi_n\|_{L^\infty}^2 \|h(u)\|_{L^2(\mu_{r_n})}^2, \end{aligned}$$

where we use the inequality $\|\psi_n\|_{L^\infty} \lesssim r_n \|\nabla_A \psi_n\|_{L^\infty}$ and the fact $r_n \leq r$ is a small radius. This gives the second of the two inequalities below, and the Sobolev Orlicz bump inequality (1.5) with bump $\Phi = \Phi_m$ as in (7.19) below gives the first one:

- (1) Orlicz-Sobolev type inequality with Φ bump and superradius φ ,

$$\Phi^{(-1)} \left(\int_{B_n} \Phi(w) d\mu_{r_n} \right) \leq C \varphi(r(B)) \int_{B_n} |\nabla_A(w)| d\mu_{r_n}, \quad w \in Lip_{\text{compact}}(B_n).$$

- (2) Cacciopoli inequality for solutions u

$$\|\nabla_A h(u)\|_{L^2(\mu_{r_{n+1}})} \leq C_n(r) \|h(u)\|_{L^2(\mu_{r_n})},$$

where

$$(4.1) \quad C(n, r) = Cn^{\frac{m-1}{2}} \|\nabla_A \psi_n\|_\infty \lesssim Cn^{2+\frac{m-1}{2}} \frac{1}{r}.$$

Taking $w = \psi_n^2 h(u)^2$ and combining the two together with $r_n = r(B_n)$ and $\gamma_n = \frac{|B(0, r_n)|}{|B(0, r_{n+1})|}$ gives

$$\begin{aligned} & \Phi^{(-1)} \left(\int_{B_{n+1}} \frac{1}{\gamma_n} \Phi(h(u)^2) d\mu_{r_{n+1}} \right) \\ & \leq C\varphi(r(B_n)) \int_{B_n} |\nabla_A (\psi_n^2 h(u)^2)| d\mu_{r_n} \\ & \leq 2C\varphi(r(B_n)) \left\{ \int_{B_n} |h(u)| |\nabla_A h(u)| d\mu_{r_n} + \int_{B_n} |\nabla_A \psi_n^2| |h(u)|^2 d\mu_{r_n} \right\} \\ & \leq 2C\varphi(r_n) \sqrt{\int_{B_n} h(u)^2 d\mu_{r_n}} \sqrt{\int_{B_n} |\nabla_A h(u)|^2 d\mu_{r_n}} + 2C(n, r)C\varphi(r_n) \|h(u)\|_{L^2(\mu_{r_n})}^2 \\ & \leq 4C(n, r)C\varphi(r_n) \|h(u)\|_{L^2(\mu_{r_n})}^2 = M(\varphi, n, r) \|h(u)\|_{L^2(\mu_{r_n})}^2, \end{aligned}$$

where

$$M(\varphi, n, r) = 4C(n, r)C\varphi(r_n).$$

Recalling the definition of $h(u) = \sqrt{\Phi^{(n)}(t^2)}$ with $\Phi = \Phi_m$ and using the submultiplicativity of Φ_m , we get

$$\begin{aligned} \int_{B(0, r_{n+1})} \Phi^{(n+1)}(u^2) d\mu_{r_{n+1}} & \leq \gamma_n \Phi \left(M(\varphi, n, r) \int_{B(0, r_n)} \Phi^{(n)}(u^2) d\mu_{r_n} \right) \\ & \leq \Phi \left(\gamma_n^* M(\varphi, n, r) \int_{B(0, r_n)} \Phi^{(n)}(u^2) d\mu_{r_n} \right), \end{aligned}$$

where $\gamma_n^* = \frac{1}{\Phi^{-1}(\frac{1}{\gamma_n})}$. Using (4.1) we can find a constant $K = K_{\text{standard}}(\varphi, r) > 1$, depending on the radius r , the superradius $\varphi(r)$, and the choice of standard sequence of Lipschitz cutoff functions $\{\psi_j\}_{j=1}^\infty$, such that

$$(4.2) \quad \gamma_n^* M(\varphi, n, r) \leq K_{\text{standard}}(\varphi, r)(n+1)^{m+1+\varepsilon},$$

which holds since we can arrange to have φ nondecreasing and $r_\infty < r_{n+1} < r_n \leq r$, $\|\nabla_A \psi_n\|_{L^\infty}^2 \leq C\frac{n^4}{r_n^2}$ and $\gamma_n = \frac{|B(0, r_n)|}{|B(0, r_{n+1})|} \leq \frac{|B(0, r_0)|}{|B(0, r_\infty)|} < \infty$, hence also $\gamma_n^* < \infty$. Therefore we have

$$(4.3) \quad \int_{B(0, r_{n+1})} \Phi^{(n+1)}(u^2) d\mu_{r_{n+1}} \leq \Phi \left(K(n+1)^{m+1+\varepsilon} \int_{B(0, r_n)} \Phi^{(n)}(u^2) d\mu_{r_n} \right).$$

Now define a sequence by

$$(4.4) \quad B_0 = \int_{B(0, r_0)} |u|^2 d\mu_{r_0}, \quad B_{n+1} = \Phi(K(n+1)^{m+1+\varepsilon} B_n).$$

The inequality (4.3) and a basic induction shows that

$$(4.5) \quad \int_{B(0, r_n)} \Phi^{(n)}(u^2) d\mu_{r_n} \leq B_n.$$

2. Iteration, maximum principle, and the Inner Ball inequality for sub solutions $u > M$

We begin this section with a weak form of the Inner Ball inequality using notation as above. Recall that $B_0 = \|u\|_{L^2(d\mu_{r_0})}^2$.

THEOREM 40. *Assume that $\varphi(r)$ and $\Phi(t) = \Phi_m(t)$ with $m > 2$ satisfy the Sobolev bump inequality (1.5), and that a standard sequence of Lipschitz cutoff functions exists. Let u be a nonnegative weak subsolution to the equation $\mathcal{L}u = \phi$ in $B(0, r)$, so that*

$$\|\phi\|_{X(B(0, r))} < e^{2^{m-1}}, \|u\|_{L^2(d\mu_r)} < e^{2^{m-1}}.$$

Then we have a constant $C(\varphi, m, r)$ determined solely by m , the radius r , and the superradius φ , such that

$$\begin{aligned} \|u\|_{L^\infty(B(0, r/2))} &\leq \sqrt{C(\varphi, m, r)}; \\ C(\varphi, m, r) &\leq \exp[C'(m)(1 + (\ln K)^m)]. \end{aligned}$$

First of all, we can assume

$$\inf_{B(0, r)} u \geq 2e^{2^{m-1}} > \|\phi\|_{X(B(0, r))}$$

by possibly replacing u with $\bar{u} \equiv u + 2e^{2^{m-1}}$ so that $2e^{2^{m-1}} \leq \|\bar{u}\|_{L^2(d\mu_r)} < 3e^{2^{m-1}}$. For convenience, we revert to writing u in place of \bar{u} for now. Applying Cacciopoli's inequality and Moser iteration as above, we obtain a sequence B_n as defined in (4.4) with its first term $4e^{2^m} \leq B_0 < 9e^{2^m}$ so that (4.5) holds. At this point we require the following two properties of the function Φ relative to the subsolution u :

$$(4.6) \quad \liminf_{n \rightarrow \infty} [\Phi^{(n)}]^{-1} \left(\int_{B(0, r_n)} \Phi^{(n)}(u^2) d\mu_{r_n} \right) \geq \|u\|_{L^\infty(\mu_{r_\infty})}^2,$$

and

$$(4.7) \quad \liminf_{n \rightarrow \infty} [\Phi^{(n)}]^{-1}(B_n) \leq C(\varphi, m, r)$$

The combination of (4.6), (4.5) and (4.7) in sequence immediately finishes the proof:

$$\|u\|_{L^\infty(\mu_{r_\infty})}^2 \leq \liminf_{n \rightarrow \infty} [\Phi^{(n)}]^{-1} \left(\int_{B(0, r_n)} \Phi^{(n)}(u^2) d\mu_{r_n} \right) \leq \liminf_{n \rightarrow \infty} [\Phi^{(n)}]^{-1}(B_n) \leq C(\varphi, m, r).$$

In order to prove the two properties (4.6) and (4.7), we need two lemmata, which are proved in the next subsection.

LEMMA 41. *Let $m > 1$. Given any $M > M_1 \geq e^{2^m}$ and $\delta \in (0, 1)$, the inequality*

$$\delta \Phi^{(n)}(M) \geq \Phi^{(n)}(M_1)$$

holds for each sufficiently large $n > N(M, M_1, \delta)$.

LEMMA 42. *Let $m > 2$, $K > 1$ and $\gamma > 0$. Consider the sequence defined by*

$$B_0 = \int_{B(0, r_0)} |u|^2 d\mu_{r_0} > e^{2^m}, \quad B_{n+1} = \Phi(K(n+1)^\gamma B_n).$$

Then there exists a positive number $C^ = C^*(B_0, K, \gamma) > M$, such that the inequality $\Phi^{(n)}(C^*) \geq B_n$ holds for each positive number n .*

It is clear that Lemma 42 proves (4.7) with the upper bound in (4.7) given by

$$C(\varphi, m, r) = C^*(9e^{2^m}, K_{\text{standard}}(\varphi, r), m + 1 + \varepsilon).$$

On the other hand, Lemma 41 implies the first property (4.6). Indeed, for any number $M_1 < \|u\|_{L^\infty(\mu_{r_\infty})}^2$, we can choose a number M so that $M_1 < M < \|u\|_{L^\infty(\mu_{r_\infty})}^2$ and let $A_M = \{x \in B(0, r_\infty) : u > \sqrt{M}\}$ whose measure is positive (recall that u is nonnegative by assumption). Without loss of generality we can assume $M_1 > e^{2^m}$ since we know $\|u\|_{L^\infty(\mu_{r_\infty})}^2 > (\inf u)^2 \geq 4e^{2^m}$. By our assumption we have

$$\int_{B(0, r_n)} \Phi^{(n)}(u^2) d\mu_{r_n} \geq \int_{A_M} \Phi^{(n)}(M) \frac{dx}{|B(0, r_n)|} \geq \frac{|A_M|}{|B(0, r)|} \Phi^{(n)}(M)$$

Thus we have

$$\liminf_{n \rightarrow \infty} [\Phi^{(n)}]^{-1} \left(\int_{B(0, r_n)} \Phi^{(n)}(u^2) d\mu_{r_n} \right) \geq \liminf_{n \rightarrow \infty} [\Phi^{(n)}]^{-1} \left(\frac{|A_M|}{|B(0, r)|} \Phi^{(n)}(M) \right) \geq M_1$$

This finishes the proof since we can take M_1 arbitrarily close to $\|u\|_{L^\infty(\mu_{r_\infty})}^2$.

Now we have an L^∞ estimate when the subsolution is relatively small in size. However, an argument based on linearity and tracking the constant $C(\varphi, m, r)$ gives the following Inner Ball inequality by applying Theorem 40 to

$$\tilde{u} \equiv \frac{u + \|\phi\|_{X(B(0, r))}}{\|u + \|\phi\|_{X(B(0, r))}\|_{L^2(d\mu_r)}} \text{ and } \tilde{\phi} \equiv \frac{\phi}{\|u + \|\phi\|_{X(B(0, r))}\|_{L^2(d\mu_r)}}$$

and noting $\mathcal{L}\tilde{u} = \tilde{\phi}$. Indeed, we then have $\|\tilde{u}\|_{L^2(d\mu_r)} = 1$ and $\|\tilde{\phi}\|_{X(B(0, r))} \leq 1$ since $u \geq 0$ implies

$$\|u + \|\phi\|_{X(B(0, r))}\|_{L^2(d\mu_r)} \geq \|\phi\|_{X(B(0, r))}\|_{L^2(d\mu_r)} = \|\phi\|_{X(B(0, r))}.$$

THEOREM 43. *Assume that $\varphi(r)$ and $\Phi(t) = \Phi_m(t)$ with $m > 2$ satisfy the (Φ, φ) -Sobolev Orlicz bump inequality (1.5), and that a standard sequence of Lipschitz cutoff functions exists. Let u be a nonnegative weak subsolution to the equation $\mathcal{L}u = \phi$ in $B(0, r)$, and suppose that $\|\phi\|_{X(B(0, r))} < \infty$. Then with $C(\varphi, m, r)$ as in Theorem 40 above, we have*

$$\|u + \|\phi\|_{X(B(0, r))}\|_{L^\infty(B(0, r/2))} \leq \sqrt{C(\varphi, m, r)} \|u + \|\phi\|_{X(B(0, r))}\|_{L^2(d\mu_r)}.$$

2.1. Abstract maximum principle. We can now obtain the analogous weak form of the maximum principle.

THEOREM 44. *Let Ω be a bounded open subset of \mathbb{R}^n . Assume that $\Phi(t) = \Phi_m(t)$ with $m > 2$ satisfies the Sobolev bump inequality for Ω . Let u be a weak subsolution to the equation $\mathcal{L}u = \phi$ in Ω and suppose that u is nonpositive on the boundary $\partial\Omega$ in the sense that $u^+ \in (W_A^{1,2})_0(\Omega)$, and suppose that $\|\phi\|_{X(\Omega)} < \infty$. Then*

$$\operatorname{esssup}_{x \in \Omega} u(x) \leq \sqrt{C(m, \Omega)} \|u + \|\phi\|_{X(\Omega)}\|_{L^2(\Omega)}.$$

PROOF. An examination of all of the arguments used to prove Theorem 43 shows that the only property we need of the cutoff functions ψ_j is that certain Sobolev and Cacciopoli inequalities hold for the functions $\psi_j h(u^+)$. But under the hypothesis $u^+ \in (W_A^{1,2})_0(\Omega)$, we can simply take $\psi_j \equiv 1$

and all of our balls B to equal Ω , since then our weak subsolution u^+ already is such that $h(u^+)$ satisfies the appropriate Sobolev and Cacciopoli inequalities. Here is a sketch of the details.

Since we may take the cutoff function ψ in the reverse Sobolev inequality in Lemma 33 to be identically 1, the Cacciopoli inequality (3.4) for a subsolution u that is nonpositive on $\partial\Omega$ now becomes simply

$$\int_{\Omega} \|\nabla_A [h(u)]\|^2 dx \leq \frac{21C_2^2}{C_1^2} \int_{\Omega} h(u)^2,$$

where the constant C_2 satisfies $C_2 \approx n^{m-1}$ for $h = \Phi_m^{(n)}$. Thus we have the following pair of inequalities for a constant $C = C(\Omega, m, n)$:

(1): Orlicz-Sobolev type inequality with Φ bump

$$\Phi^{(-1)} \left(\int_{\Omega} \Phi(w) dx \right) \leq C \int_{\Omega} |\nabla_A(w)| dx, \quad w \in Lip_{\text{compact}}(\Omega).$$

(2): Cacciopoli inequality for subsolutions u that are nonpositive on $\partial\Omega$,

$$\|\nabla_A h(u)\|_{L^2(\Omega)} \leq C \|h(u)\|_{L^2(\Omega)}.$$

Taking $w = h(u)^2$ and combining the two together gives

$$\begin{aligned} & \Phi^{(-1)} \left(\int_{\Omega} \Phi(h(u)^2) dx \right) \\ & \leq C \int_{\Omega} |\nabla_A(h(u)^2)| dx = 2C \left\{ \int_{\Omega} |h(u)| |\nabla_A h(u)| dx \right\} \\ & \leq 2C \sqrt{\int_{\Omega} h(u)^2 dx} \sqrt{\int_{\Omega} |\nabla_A h(u)|^2 dx} \leq 2C^2 \|h(u)\|_{L^2(\Omega)}^2. \end{aligned}$$

Recalling the definition of $h(u) = \sqrt{\Phi^{(n)}(t^2)}$ with $\Phi = \Phi_m$ we get,

$$(4.8) \quad \int_{\Omega} \Phi^{(n+1)}(u^2) dx \leq \Phi \left(C \int_{\Omega} \Phi^{(n)}(u^2) dx \right).$$

Now we proceed exactly as above to complete the proof. ■

At this point we wish to replace the right hand side above by $C \|\phi\|_{X(\Omega)}$, and here we will follow an argument of Gutierrez and Lanconelli [GuLa]. Recall that $u \in W_A^{1,2}(\Omega)$ is a weak subsolution of

$$(4.9) \quad Lu \equiv \nabla^{\text{tr}} \mathcal{A}(x, u(x)) \nabla u = \phi,$$

if

$$(4.10) \quad - \int_{\Omega} \nabla u^{\text{tr}} A \nabla w \geq \int_{\Omega} \phi w$$

for all nonnegative $w \in \left(W_A^{1,2}\right)_0(\Omega)$. Now let $\tilde{u} = h \circ u$, where h is increasing and piecewise continuously differentiable on $[0, \infty)$. Then \tilde{u} formally satisfies the equation

$$\mathcal{L}\tilde{u} = \nabla^{\text{tr}} A \nabla (h \circ u) = \nabla^{\text{tr}} A h'(u) \nabla u = h'(u) \mathcal{L}u + h''(u) (\nabla u)^{\text{tr}} A \nabla u,$$

and if u is a positive subsolution of $Lu = \phi$ in Ω , we have

$$(4.11) \quad \begin{aligned} - \int (\nabla w)^{\text{tr}} A \nabla \tilde{u} &= \int w \mathcal{L} \tilde{u} = \int w h'(u) \mathcal{L} u + \int w h''(u) \|\nabla_A u\|^2 \\ &\geq \int w h'(u) \phi + \int w h''(u) \|\nabla_A u\|^2, \end{aligned}$$

provided $wh'(u)$ is nonnegative and in the space $\left(W_A^{1,2}\right)_0(\Omega)$, which will be the case if in addition h' is bounded.

THEOREM 45. *Let Ω be a bounded open subset of \mathbb{R}^n . Let u be a weak subsolution of (4.9) with ϕ A -admissible, i.e. $\|\phi\|_{X(\Omega)} < \infty$. Then the following maximum principle holds,*

$$(4.12) \quad \sup_{\Omega} u \leq \sup_{\partial\Omega} u + C \|\phi\|_{X(\Omega)},$$

where the constant C depends only on Ω .

PROOF. We first suppose that in addition we have $u \in \left(W_A^{1,2}\right)_0(B(0,r))$, so that $\sup_{\partial\Omega} u = 0$. The proof basically repeats the proof of Step 2 of Theorem 3.1 in [GuLa] but we repeat it here for convenience. We may assume that u has been replaced with $u^+ = \max\{u, 0\}$. So let u be a nonnegative weak subsolution to (4.9). By Theorem 44 we have that u satisfies a global boundedness inequality

$$\|u\|_{L^\infty(\Omega)} \leq C \left(\|u\|_{L^2(\Omega)} + \|\phi\|_{X(\Omega)} \right)$$

Now denote $M \equiv \text{esssup}_{\Omega} u$, $\kappa \equiv \|\phi\|_{X(\Omega)}$ and consider $w = \frac{u}{M+\kappa-u}$. It is easy to see that $w \in \left(W_A^{1,2}\right)_0(\Omega)$ and substituting in (4.10) we obtain

$$- \int_{\Omega} \frac{\nabla u^{\text{tr}} A \nabla u}{(M+\kappa-u)^2} \geq \int_{\Omega} \frac{u \phi}{M+\kappa-u}$$

Dividing by $M+\kappa$ and using that $u \leq M+\kappa$ we claim that

$$(4.13) \quad \int_{\Omega} \frac{|\nabla_A u|^2}{(M+\kappa-u)^2} \leq 8|\Omega| = C(\Omega).$$

Indeed, we have from $\kappa \equiv \|\phi\|_{X(\Omega)}$ and the definition of the norm in $X(\Omega)$,

$$\begin{aligned} \int_{\Omega} \frac{|\nabla_A u|^2}{(M+\kappa-u)^2} &\leq \int_{\Omega} |\phi| \frac{u}{(M+\kappa)(M+\kappa-u)} \\ &\leq \frac{1}{M+\kappa} \|\phi\|_{X(\Omega)} \int_{\Omega} \left| \nabla_A \frac{u}{M+\kappa-u} \right| \\ &= \kappa \int_{\Omega} \frac{|\nabla_A u|}{(M+\kappa-u)^2} \leq \int_{\Omega} \frac{|\nabla_A u|}{M+\kappa-u} \\ &\leq \frac{1}{2} \int_{\Omega} \frac{|\nabla_A u|^2}{(M+\kappa-u)^2} + 4|\Omega|. \end{aligned}$$

Now define

$$h(t) = \begin{cases} \log \frac{M+\kappa}{M+\kappa-t} & \text{for } t \leq M \\ \log \frac{M+\kappa}{\kappa} & \text{for } t > M \end{cases}$$

It is easy to calculate that for $t \leq M$ we have

$$h'(t) = \frac{1}{M + \kappa - t}$$

$$h''(t) = \frac{1}{(M + \kappa - t)^2}$$

and therefore we have from (4.13) for $\tilde{u} \equiv h(u)$ the estimate

$$(4.14) \quad \int \|\nabla_A \tilde{u}\|^2 \leq C(m, \Omega).$$

Now we would like to obtain an equation that $\tilde{u} = h(u)$ satisfies. Substituting h in (4.11) we have

$$- \int (\nabla w)^{\text{tr}} A \nabla \tilde{u} \geq \int \frac{w\phi}{M + \kappa - u} + \int \frac{w}{(M + \kappa - t)^2} \|\nabla_A u\|^2 \geq \int \frac{w\phi}{M + \kappa - u},$$

since w is nonnegative. Therefore, \tilde{u} is a nonnegative weak subsolution of $L\tilde{u} = \phi/(M + \kappa - u)$. Moreover, since $u = 0$ on $\partial\Omega$ and consequently $\tilde{u} = 0$ on $\partial\Omega$, we have that \tilde{u} satisfies the global boundedness inequality

$$\|\tilde{u}\|_{L^\infty(\Omega)} \leq C \left(\|\tilde{u}\|_{L^2(\Omega)} + \left\| \frac{\phi}{M + \kappa - u} \right\|_{X(\Omega)} \right)$$

For the last term on the right we use the monotonicity property $\|f\|_{X(\Omega)} \leq \|g\|_{X(\Omega)}$ if $|f| \leq |g|$ to obtain¹

$$\left\| \frac{\phi}{M + \kappa - u} \right\|_{X(\Omega)} \leq \left\| \frac{\phi}{\kappa} \right\|_{X(\Omega)} = 1,$$

while for the first term on the right we have from Sobolev inequality and (4.14),

$$\int \|\tilde{u}\|^2 \leq C(\Omega) \int \|\nabla_A \tilde{u}\|^2 \leq C(\Omega).$$

Combining the above gives

$$\|\tilde{u}\|_{L^\infty(\Omega)} \leq C(\Omega),$$

and recalling the definition of $\tilde{u} = h(u)$ gives

$$M + \kappa \leq (M + \kappa - u)e^{C(\Omega)},$$

$$M \leq \kappa(e^{C(\Omega)} - 1)$$

Recalling the definitions $M \equiv \sup_\Omega u$, $\kappa \equiv \|\phi\|_{X(\Omega)}$ we conclude that (4.12) holds in the case $\sup_{\partial\Omega} u = 0$.

To handle the general case define $\bar{u} \equiv (u - \sup_{\partial\Omega} u)^+$. Then \bar{u} is a nonnegative weak subsolution of $L\bar{u} = -|\phi|$ and $\bar{u} = 0$ on $\partial\Omega$, therefore Theorem 45 applies and the estimate above follows from (4.12). ■

2.2. Proof of Recurrence Inequalities.

¹This is the only place the monotonicity of the norm $\|\cdot\|_{X(\Omega)}$ is used, and in the rest of the paper, we could use instead the larger space $X(\Omega)$ in which absolute values appear *outside* the integral in the numerator of the definition of the norm $\|\cdot\|_{X(\Omega)}$.

Proof of Lemma 41. This is straightforward since we know $\Phi^{(n)}(t) = e^{(n+(\ln t)^{1/m})^m}$. We can use the notation $a = (\ln M)^{1/m} > (\ln M_1)^{1/m} = b$ and obtain

$$\delta\Phi^{(n)}(M) \geq \Phi^{(n)}(M_1) \iff \delta e^{(n+a)^m} > e^{(n+b)^m} \iff \ln \delta + (n+a)^m > (n+b)^m$$

This is always true when n is sufficiently large, because if $m > 1$, we have

$$\lim_{n \rightarrow \infty} [(n+a)^m - (n+b)^m] \geq \lim_{n \rightarrow \infty} (a-b) \cdot m(b+n)^{m-1} = \infty.$$

Proof of Lemma 42. Let us define another sequence by

$$A_0 = C^*, \quad A_{n+1} = \Phi(A_n), \quad n \geq 0$$

Thus we are trying to find a number C such that $A_n \geq B_n$ holds for all $n \geq 0$. Next we pass to another two sequences:

$$\begin{aligned} a_n &= (\ln A_n)^{1/m}, \\ b_n &= (\ln B_n)^{1/m}. \end{aligned}$$

The sequence $\{a_n\}$ satisfies $a_0 = (\ln C^*)^{1/m}$ and

$$a_n = (\ln A_n)^{1/m} = (\ln \Phi(A_{n-1}))^{1/m} = (\ln A_{n-1})^{1/m} + 1 = a_{n-1} + 1$$

As for the other sequence, it is clear that $b_0 = (\ln B_0)^{1/m} > 2$, but the recurrence relation for b_n is a bit more complicated, and with $K = K_{\text{standard}}(r)$ we have:

$$\begin{aligned} b_n &= (\ln B_n)^{1/m} = (\ln \Phi(Kn^\gamma B_{n-1}))^{1/m} = (\ln(Kn^\gamma B_{n-1}))^{1/m} + 1 \\ &= (b_{n-1}^m + \ln(Kn^\gamma))^{1/m} + 1. \end{aligned}$$

This is clear that $b_n > b_{n-1} + 1$ thus we have a rough lower bound $b_n \geq n + b_0$. Since the function $g(x) = x^{1/m}$ is concave, we have

$$b_n = \{b_{n-1}^m + \ln(Kn^\gamma)\}^{1/m} + 1 = b_{n-1} \left\{ 1 + \frac{\ln(Kn^\gamma)}{b_{n-1}^m} \right\}^{1/m} + 1 \leq b_{n-1} + \frac{\ln(Kn^\gamma)}{m \cdot b_{n-1}^{m-1}} + 1$$

Thus

$$b_n \leq b_0 + n + \frac{1}{m} \sum_{j=1}^n \frac{\ln(Kj^\gamma)}{b_{j-1}^{m-1}} \implies a_n - b_n \geq a_0 - b_0 - \frac{1}{m} \sum_{j=1}^n \frac{\ln(Kj^\gamma)}{b_{j-1}^{m-1}}$$

Because $m > 2$, we have

$$\sum_{j=1}^{\infty} \frac{\ln(Kj^\gamma)}{b_{j-1}^{m-1}} < \sum_{j=1}^{\infty} \frac{\ln(Kj^\gamma)}{(b_0 + j - 1)^{m-1}} \leq C(m)b_0^{2-m}(\gamma + \ln K) < \infty,$$

and we can choose $a_0 = b_0 + C(m)b_0^{2-m}(\gamma + \ln K)$ and guarantee $a_n > b_n$ for all $n \geq 0$. The choice of $C^* = C^*(B_0, K, \gamma)$ is

$$(4.15) \quad C^* = \exp(a_0^m) \leq \exp\left(b_0 + \frac{C(\gamma + \ln K_{\text{standard}}(r))}{(m-2)b_0^{m-2}}\right)^m, \quad b_0 = (\ln B_0)^{\frac{1}{m}}.$$

Thus we have the estimate

$$\begin{aligned} (4.16) \quad C(m, r) &= C^*(9e^{2^m}, K_{\text{standard}}(r), m+1+\varepsilon) \\ &\leq \exp\left(\left[\ln(9e^{2^m})\right]^{\frac{1}{m}} + C \frac{m+1+\varepsilon + \ln K_{\text{standard}}(r)}{(\ln B_0)^{m-2}}\right)^m. \end{aligned}$$

REMARK 46. Lemma 42 fails for $m = 2$ even with $\gamma = 0$ and $K > e$. Indeed, then from the calculations above we have

$$b_n = b_{n-1} \left\{ 1 + \frac{\ln K}{b_{n-1}^2} \right\}^{1/2} + 1 \geq b_{n-1} + \frac{\ln K}{4b_{n-1}} + 1, \quad \text{for } n \text{ large,}$$

which when iterated gives

$$b_n \geq b_0 + n + \sum_{j=0}^{n-1} \frac{\ln K}{4b_j} = b_0 + n + \frac{\ln K}{4} \sum_{j=0}^{n-1} \frac{1}{b_j}.$$

So if there are positive constants A, B such that $b_n \leq An + B$ for n large, then we would have

$$b_n \geq b_0 + n + \frac{\ln K}{4} c \ln n$$

for some positive constant c , which is a contradiction. Thus $b_n \leq a_0 + n$ for all $n \geq 1$ is impossible. Moreover we have

$$\Phi^{(-n)}(B_n) = e^{\left[(\ln B_n)^{\frac{1}{m}} - n \right]^m} = e^{[b_n - n]^m} \geq e^{[b_0 + \frac{\ln K}{4} c \ln n - n]^m} \nearrow \infty$$

as $n \rightarrow \infty$, so that the left hand side of (4.7) is infinite.

CHAPTER 5

Continuity of weak solutions u

In this final chapter of Part 2 of the paper, we turn first to establishing a Harnack inequality, and for this we will adapt an argument of Bombieri (see [Mos, Lemma 3]). One needs to be careful however, since in this case the coefficients in the inequalities depend on the radius r of the ball. Moreover, the constant $C_{Har}(r)$ in the Harnack inequality we obtain will depend on the complicated constants in the Inner Ball inequalities, which as we will see later, typically blow up as $r \rightarrow 0$ when the underlying geometry *fails* to be of finite type. Finally, we need to carefully define our bump function $\Phi(t)$ for small values of t rather than large values as above. Then we give an affine extension for large values of t that results in a bump function that is supermultiplicative rather than submultiplicative.

Recall that the basic idea in Bombieri iteration is to implement a sequence of iterations, but in the reverse direction of Moser iterations. It might be of some help to indicate the geometry used here by describing the radii involved in this "iteration within an iteration". Let us fix attention on a ball $B(0, 1)$ (in some metric space) of radius 1 centered at the origin. Then Bombieri considers an *increasing* sequence of subballs $B(0, \nu_j)$ with radii

$$\frac{1}{2} = \nu_0 < \nu_1 < \nu_2 < \dots < \nu_j < \nu_{j+1} < \dots \nearrow 1$$

defined by $\nu_{j+1} - \nu_j = \frac{1}{8} \left(\frac{3}{4}\right)^j$. Within each annulus $B(0, \nu_{j+1}) \setminus B(0, \nu_j)$, Bombieri performs a Moser iteration to obtain a generalized Inner Ball inequality beginning with the larger ball $B(0, \nu_{j+1})$ and ending at the smaller ball $B(0, \nu_j)$ by constructing a *decreasing* sequence of balls $B(0, r_k^j)$ with radii

$$\nu_{j+1} = r_0^j > r_1^j > r_2^j > \dots > r_k^j > r_{k+1}^j \dots \searrow \nu_j$$

defined by $r_k^j - r_{k+1}^j = (\nu_{j+1} - \nu_j) \frac{6}{\pi^2} \left(\frac{1}{k+1}\right)^2 = \frac{2}{\pi^2} \left(\frac{3}{4}\right)^{j+1} \left(\frac{1}{k+1}\right)^2$. Then a delicate use of the sequence of these Inner Ball inequalities controls from above the supremum of u by an exponential of an average of $\ln u$ instead of by an L^2 norm of u . Then a similar construction is carried out with $\frac{1}{u}$ in place of u that controls from below the infimum of u by an exponential of the same average of $\ln u$. As a consequence of these two estimates $\sup u \lesssim \exp CAvg(\ln u)$ and $\inf u \gtrsim \exp CAvg(\ln u)$, we obtain a strong Harnack inequality $\sup u \lesssim \inf u$.

We now turn to the details of adapting the Bombieri argument to our situation where the constants in our Inner Ball inequality exhibit greater blowup in the infinitely degenerate situation than in the classical or finite type cases. It is useful to begin by noting that the constant $K = K_{\text{nonstandard}}(\varphi, r_0, \nu)$ in our Inner Ball inequality for the more general annulus $B(0, r_0) \setminus B(0, \nu r_0)$ satisfies the estimate

$$K = K_{\text{nonstandard}}(\varphi, r_0, \nu) = \frac{C\varphi(\nu r_0)}{(1 - \nu)\delta(\nu r_0)}$$

and so from the Inner Ball Inequalities proved below we obtain a bound

$$\|h(u)\|_{L^\infty(\nu B_0)} \leq \sqrt{C(\varphi, m, r, \nu)} \|u\|_{L^2(B_0)}$$

for all subsolutions u and appropriate nonlinear functions h . It will turn out to be important that the dependence on $\frac{1}{1-\nu}$ is subexponential rather than exponential.

Continuity will be derived from this later on by using an argument of DeGiorgi.

1. Bombieri and half Harnack for a reciprocal of a solution $u < \frac{1}{M}$

We first consider the reciprocal of a positive bounded weak solution.

1.1. Moser iteration for negative powers of a positive bounded solution. Here we assume the inhomogeneous Sobolev bump inequality (7.14) holds. Let us start by fixing $r > 0$ and recalling the *nonstandard* sequence of Lipschitz cutoff functions $\{\psi_j\}_{j=1}^\infty$ depending on r , along with the sets $B(0, r_j) \supset \text{supp } \psi_j$ for which $\psi_j = 1$ on $B(0, r_{j+1})$ as given in Definition 22. Without loss of generality we are considering only balls $B(0, r)$ centered at the origin here, since it is only on the x_2 -axis that continuity is in doubt.

We apply Lemma 34 with $h_\beta(t) = \sqrt{\Phi_m^{(n)}(t^{2\beta})} = h(t^\beta)$ where $h(s) \equiv \sqrt{\Phi_m^{(n)}(s^2)}$, and where $-\frac{1}{2} < \beta < 0$, and $\psi = \psi_n$ and obtain

$$\int_{B(0, r_n)} \psi_n^2 \|\nabla_A [h(u^\beta)]\|^2 d\mu_{r_n} \leq C(n+1)^{m-1} \int_{B(0, r_n)} [h(u^\beta)]^2 (|\nabla_A \psi_n|^2 + \psi_n^2) d\mu_{r_n}$$

This implies

$$\begin{aligned} \|\nabla_A [\psi_n h(u^\beta)]\|_{L^2(\mu_{r_n})}^2 &\leq 2 \|\psi_n \nabla_A h(u^\beta)\|_{L^2(\mu_{r_n})}^2 + 2 \|\nabla_A \psi_n |h(u^\beta)|\|_{L^2(\mu_{r_n})}^2 \\ &\leq C(n+1)^{m-1} \int_{B(0, r_n)} [h(u^\beta)]^2 (|\nabla_A \psi_n|^2 + \psi_n^2) d\mu_{r_n} + 2 \|\nabla_A \psi_n |h(u^\beta)|\|_{L^2(\mu_{r_n})}^2 \\ &\leq C(n+1)^{m-1} \|\nabla_A \psi_n\|_{L^\infty}^2 \|h(u^\beta)\|_{L^2(\mu_{r_n})}^2, \end{aligned}$$

where we use the inequality $\|\psi_n\|_{L^\infty} \lesssim r_n \|\nabla_A \psi_n\|_{L^\infty}$ and the fact $r_n \leq r$ is a small radius. This gives the second of the two inequalities below, and the Sobolev inequality (1.5) with bump Φ gives the first one:

- (1) Orlicz-Sobolev type inequality with Φ bump and superradius φ ,

$$\Phi^{(-1)} \left(\int_{B_n} \Phi(w) d\mu_{r_n} \right) \leq C\varphi(r(B_n)) \int_{B_n} |\nabla_A(w)| d\mu_{r_n}, \quad u \in Lip_{\text{compact}}(B)$$

- (2) Cacciopoli inequality for solutions u

$$\|\nabla_A h(u^\beta)\|_{L^2(\mu_{r_{n+1}})} \leq C(n, r) \|h(u^\beta)\|_{L^2(\mu_{r_n})}$$

where

$$(5.1) \quad C(n, r, \nu) = Cn^{\frac{m-1}{2}} \|\nabla_A \psi_n\|_\infty \leq \frac{Cn^{2+\frac{m-1}{2}}}{(1-\nu)\delta(r_n)}.$$

Taking $w = \psi_n^2 h(u)^2$ and combining the two together gives

$$\begin{aligned}
& \Phi^{(-1)} \left(\int_{B_{n+1}} \Phi(h(u^\beta)^2) d\mu_{r_n} \right) \\
& \leq C\varphi(r(B_n)) \int_{B_n} |\nabla_A (\psi_n^2 h(u^\beta)^2)| d\mu_{r_n} \\
& \leq 2C\varphi(r(B_n)) \left\{ \int_{B_n} |h(u^\beta)| |\nabla_A h(u^\beta)| d\mu_{r_n} + \int_{B_n} |h(u^\beta)|^2 |\nabla_A \psi_n^2| d\mu_{r_n} \right\} \\
& \leq 2C\varphi(r_n) \sqrt{\int_{B_n} h(u^\beta)^2 d\mu_{r_n}} \sqrt{\int_{B_n} |\nabla_A h(u^\beta)|^2 d\mu_{r_n}} + 2C(n, r, \nu) C\varphi(r_n) \|h(u^\beta)\|_{L^2(\mu_{r_n})}^2 \\
& \leq 4C(n, r, \nu) C\varphi(r_n) \|h(u^\beta)\|_{L^2(\mu_{r_n})}^2 = M(\varphi, n, r, \nu) \|h(u^\beta)\|_{L^2(\mu_{r_n})}^2,
\end{aligned}$$

where

$$M(\varphi, n, r, \nu) = 4C(n, r, \nu) C\varphi(r_n).$$

Recalling the definition of $h(u^\beta) = \sqrt{\Phi_m^{(n)}(u^{2\beta})}$ and $\frac{|B(0, r_n)|}{|B(0, r_{n+1})|} \leq \frac{1}{2}$ we get,

$$\int_{B(0, r_{n+1})} \Phi^{(n+1)}(u^{2\beta}) d\mu_{r_{n+1}} \leq \Phi \left(M(\varphi, n, r, \nu) \int_{B(0, r_n)} \Phi^{(n)}(u^{2\beta}) d\mu_{r_n} \right).$$

Using (5.1), we see that

$$M(\varphi, n, r, \nu) \leq K_{\text{nonstandard}}(\varphi, r, \nu) n^\gamma, \quad \gamma = 2 + \frac{m-1}{2},$$

where

$$(5.2) \quad K = K_{\text{nonstandard}}(\varphi, r, \nu) \equiv \frac{C\varphi(\nu r)}{(1-\nu)\delta(\nu r)},$$

since $\frac{r}{\delta(r)}$ is nonincreasing. Therefore we have

$$\int_{B(0, r_{n+1})} \Phi^{(n+1)}(u^{2\beta}) d\mu_{r_{n+1}} \leq \Phi \left(K n^\gamma \int_{B(0, r_n)} \Phi^{(n)}(u^{2\beta}) d\mu_{r_n} \right).$$

Now let us define a sequence by

$$(5.3) \quad B_0 = \int_{B(0, r_0)} |u|^{2\beta} d\mu_{r_0}, \quad B_{n+1} = \Phi(K n^\gamma B_n).$$

The inequality (4.3) and a basic induction shows

$$(5.4) \quad \int_{B(0, r_n)} \Phi^{(n)}(u^{2\beta}) d\mu_{r_n} \leq B_n.$$

1.2. Iteration and the Inner Ball inequality for sub solutions u^β with $\beta < 0$. Now we continue with a weak form of the Inner Ball inequality analogous to Theorem 40, but for nonnegative bounded weak *solutions*.

THEOREM 47. Assume that $\varphi(r)$ and $\Phi(t) = \Phi_m(t)$ with $m > 2$ satisfy the Sobolev bump inequality (1.5), and that a nonstandard sequence of Lipschitz cutoff functions exists. Let $\beta < 0$ and u be a nonnegative bounded weak solution to the equation $\mathcal{L}u = \phi$ in $B(0, r)$, so that

$$\|\phi\|_{X(B(0, r))} < e^{2^{m-1}}, \|u^\beta\|_{L^2(d\mu_r)} < e^{2^{m-1}}.$$

Then we have a constant $C(\varphi, m, r, \nu)$ determined by m , the radius r and the geometry, such that

$$\begin{aligned} \|u^\beta\|_{L^\infty(B(0, \nu r))} &\leq \sqrt{C(\varphi, m, r, \nu)}; \\ C(\varphi, m, r, \nu) &\leq \exp\{C'(m)(1 + \ln K)^m\}, \quad C'(m) < \infty \text{ for } m > 2; \\ K &= K_{\text{nonstandard}}(\varphi, r, \nu) = \frac{C\varphi(\nu r)}{(1 - \nu)\delta(\nu r)}. \end{aligned}$$

First of all, since $\beta < 0$, we can assume

$$\inf_{B(0, r)} u^\beta \geq 2e^{2^{m-1}} > \|\phi\|_{X(B(0, r))},$$

by rescaling u and ϕ by a controlled constant, so that after rescaling we have

$$2e^{2^{m-1}} \leq \|u^\beta\|_{L^2(d\mu_r)} \leq Ce^{2^{m-1}}.$$

Thus we have $B_0 = \int_{B(0, r_0)} |u|^{2\beta} d\mu_{r_0} \approx e^{2^m}$. Applying Cacciopoli's inequality and Moser iteration, we obtain a sequence B_n as defined in (5.3) with its first term $4e^{2^m} < B_0 < 9e^{2^m}$ so that (5.4) holds. At this point we require the following two properties of the function Φ relative to the solution u :

$$(5.5) \quad \liminf_{n \rightarrow \infty} [\Phi^{(n)}]^{-1} \left(\int_{B(0, r_n)} \Phi^{(n)}(u^{2\beta}) d\mu_{r_n} \right) \geq \|u^\beta\|_{L^\infty(\mu_{r_\infty})}^2,$$

and

$$(5.6) \quad \liminf_{n \rightarrow \infty} [\Phi^{(n)}]^{-1}(B_n) \leq C(\varphi, m, r, \nu)$$

The combination of (5.5), (5.4) and (5.6) in sequence immediately finishes the proof:

$$\|u^\beta\|_{L^\infty(\mu_{r_\infty})}^2 \leq \liminf_{n \rightarrow \infty} [\Phi^{(n)}]^{-1} \left(\int_{B(0, r_n)} \Phi^{(n)}(u^{2\beta}) d\mu_{r_n} \right) \leq \liminf_{n \rightarrow \infty} [\Phi^{(n)}]^{-1}(B_n) \leq C(\varphi, m, r, \nu).$$

The two properties (5.5) and (5.6) are now proved just as in Section 2, where we used Lemmas 41 and 42 there. For future reference we record the analogue, for our situation here, of the bound for the constant C^* in (4.15) arising in the proof of Lemma 42:

$$\begin{aligned} (5.7) \quad C^* &= \exp(a_0^m) = \exp\left(b_0 + \frac{C(m)(\gamma + \ln K)}{b_0^{m-2}}\right)^m \\ &\leq \exp\left[C'(m) \left(\ln B_0 + \frac{(\gamma + \ln K)^m}{b_0^{(m-2)m}}\right)\right] \\ &\leq e^{C'(m)(1 + \ln K)^m}, \end{aligned}$$

where the final inequality follows since $\ln B_0 \approx 2^m$ is controlled. The constant K is now larger than before, namely

$$K = \frac{C\varphi(\nu r)}{(1 - \nu)\delta(\nu r)}.$$

Just as in the previous section, an argument based on linearity gives

THEOREM 48. *Assume that $\varphi(r)$ and $\Phi(t) = \Phi_m(t)$ with $m > 2$ satisfy the Sobolev bump inequality (1.5), and that a nonstandard sequence of Lipschitz cutoff functions exists. Let u be a nonnegative bounded weak solution to the equation $\mathcal{L}u = \phi$ in $B(0, r)$, so that $\|\phi\|_{X(B(0, r))} < \infty$. Then with $C(\varphi, m, r, \nu)$ as in Theorem 47 above we have*

$$\begin{aligned} \|(u + \phi)_{X(B(0, r))}\|^\beta_{L^\infty(B(0, r/2))} &\leq \sqrt{C(m, r)} \left(\|(u + \phi)_{X(B(0, r))}\|^\beta_{L^2(d\mu_r)} \right), \\ C(\varphi, m, r, \nu) &\leq \exp[C'(m)(1 + \ln K)^m]. \end{aligned}$$

PROOF. Given u we set $\tilde{u} = \frac{u + \|\phi\|_{X(B(0, r))}}{\|(u + \|\phi\|_{X(B(0, r))})^\beta_{L^2(d\mu_r)}^{\frac{1}{\beta}}}$ and $\tilde{\phi} = \frac{\phi}{\|(u + \|\phi\|_{X(B(0, r))})^\beta_{L^2(d\mu_r)}^{\frac{1}{\beta}}}$. Now apply Theorem 47 to \tilde{u}^β to get

$$\|\tilde{u}^\beta\|_{L^\infty(B(0, r/2))} \leq \sqrt{C(\varphi, m, r, \nu)},$$

which is

$$\|(u + \|\phi\|_{X(B(0, r))})^\beta_{L^\infty(B(0, r/2))} \leq \sqrt{C(\varphi, m, r, \nu)} \left\| (u + \|\phi\|_{X(B(0, r))})^\beta \right\|_{L^2(d\mu_r)}.$$

■

1.3. Bombieri's lemma for sub solutions u^β with $\beta < 0$. To handle negative powers of the solution we will use the following adaptation of a lemma of Bombieri. In our application below we will substitute $w = \frac{1}{u}$ where u is a weak solution, so that with $\theta \equiv \beta > 0$ we have that $w^\beta = u^{-\beta}$ is a weak subsolution to which our Inner Ball inequality applies.

LEMMA 49. *Let $1 \leq w < \infty$ be a measurable function defined in a neighborhood of a ball $B(y_0, r_0)$. Suppose there exist positive constants τ , A , and $0 < \nu_0 < 1$, $a \geq 0$; and locally bounded functions $c_1(y, r)$, $c_2(y, r)$, with $1 \leq c_1(y, r), c_2(y, r) < \infty$ for all $0 < r < \infty$, such that for all $B(y, r) \subset B(y_0, r_0)$ the following two conditions hold*

(1)

$$(5.8) \quad \operatorname{esssup}_{x \in \nu B(y, r)} w^\theta \leq c_1(y, \nu r) e^{A(\ln \frac{1}{1-\nu})^\tau} \left\{ \frac{1}{|B(y, r)|} \int_{B(y, r)} w^{2\theta} \right\}^{1/2}$$

for every $0 < \nu_0 \leq \nu < 1$, $0 < \theta < 1/2$, and

(2)

$$(5.9) \quad s |\{x \in B(y, r) : \log w > s + a\}| < c_2(y, r) |B(y, r)|$$

for every $s > 0$.

Then, for every ν with $0 < \nu_0 \leq \nu < 1$ there exists $b = b(\nu, \tau, c_1, c_2)$ such that

$$(5.10) \quad \operatorname{esssup}_{B(y, \nu r)} w < b e^a.$$

More precisely, b is given by

$$b = \exp(C(\nu_0, A, \tau) c_2^*(r) c_1^*(r)),$$

where

$$(5.11) \quad c_j^* = c_j^*(y, r, \nu) = \operatorname{esssup}_{\nu \leq s \leq 1} c_j(y, sr), \quad j = 1, 2.$$

and the constant $C(\nu_0, A, \tau)$ is bounded for ν_0 away from 1.

PROOF. Fix $\nu_0 \leq \nu < 1$. Define

$$\Omega(\rho) = \operatorname{esssup}_{x \in B(y, \rho)} (\log w(y) - a) \quad \text{for } \nu r \leq \rho \leq r.$$

First, note that if $\Omega(\nu r) \leq 0$ then estimate (5.10) holds with any $b > 1$, therefore, we may assume $\Omega(\rho) > 0$ for all $\nu r \leq \rho \leq r$. We then decompose the ball $B = B(y, r)$ in the following way

$$\begin{aligned} B &= B_1 \cup B_2 \\ &= \left\{ y \in B : \log w(y) - a > \frac{1}{2} \Omega(r) \right\} \cup \left\{ y \in B : \log w(y) - a \leq \frac{1}{2} \Omega(r) \right\}. \end{aligned}$$

For simplicity, we will write $c_i(y, r) = c_i(r)$, $i = 1, 2$. We then have

$$\begin{aligned} e^{-2\theta a} \int_B w^{2\theta} &= \int_B \exp(2\theta(\log w - a)) \\ &\leq \int_{B_1} \exp(2\theta\Omega(r)) + \int_{B_2} \exp(\theta\Omega(r)), \end{aligned}$$

since $2\theta(\log w - a) \leq 2\theta\Omega(r)$ by definition of Ω , and since $y \in B_2$ implies that $2\theta(\log w - a) \leq 2\theta\frac{1}{2}\Omega(r)$. Thus by condition (5.9) we get

$$e^{-2\theta a} \int_B w^{2\theta} \leq e^{2\theta\Omega(r)} \frac{2c_2(r)}{\Omega(r)} |B| + e^{\theta\Omega(r)} |B|,$$

and hence

$$(5.12) \quad \frac{e^{-2\theta a}}{|B|} \int_B w^{2\theta} \leq \frac{2c_2(r)}{\Omega(r)} e^{2\theta\Omega(r)} + e^{\theta\Omega(r)}.$$

Since $\theta\Omega(\nu r) = \log \operatorname{esssup}_{x \in \nu B} w^\theta - \theta a$, we have, using first (5.8) and then (5.12), that

$$\begin{aligned} \Omega(\nu r) &= \frac{1}{\theta} \log \operatorname{esssup}_{x \in \nu B} w^\theta - a \leq \frac{1}{\theta} \log \left(c_1(\nu r) e^{A(\ln \frac{1}{1-\nu})^\tau} \left\{ \frac{1}{|B|} \int_B w^{2\theta} \right\}^{1/2} \right) - a \\ (5.13) \quad &\leq \frac{1}{\theta} \log \left(c_1(\nu r) e^{A(\ln \frac{1}{1-\nu})^\tau} \right) + \frac{1}{2\theta} \log \left(\left\{ \frac{1}{|B|} \int_B w^{2\theta} \right\} e^{-2\theta a} \right) \end{aligned}$$

$$(5.14) \quad \leq \frac{1}{\theta} \log \left(c_1(\nu r) e^{A(\ln \frac{1}{1-\nu})^\tau} \right) + \frac{1}{2\theta} \log \left(\frac{2c_2(r)}{\Omega(r)} e^{2\theta\Omega(r)} + e^{\theta\Omega(r)} \right).$$

Consider first the case

$$(5.15) \quad 0 < \frac{1}{\Omega(r)} \log \left(\frac{\Omega(r)}{2c_2(r)} \right) < 1/2.$$

In this case we can choose $\theta = \frac{1}{\Omega(r)} \log \left(\frac{\Omega(r)}{2c_2(r)} \right)$, and then the two terms in brackets on the right-hand side of (5.14) are equal, i.e. $(2c_2(r)/\Omega(r)) e^{2\theta\Omega(r)} = e^{\theta\Omega(r)}$, and so

$$\begin{aligned} \Omega(\nu r) &\leq \frac{1}{\theta} \log \left(c_1(\nu r) e^{A \left(\ln \frac{1}{1-\nu} \right)^\tau} \right) + \frac{1}{2\theta} \log \left(2e^{\theta\Omega(r)} \right) \\ &= \frac{1}{\theta} \log \left(\sqrt{2}c_1(\nu r) e^{A \left(\ln \frac{1}{1-\nu} \right)^\tau} \right) + \frac{1}{2} \Omega(r) \\ &= \left(\frac{\log \left(\sqrt{2}c_1(\nu r) e^{A \left(\ln \frac{1}{1-\nu} \right)^\tau} \right)}{\log \left(\frac{\Omega(r)}{2c_2(r)} \right)} + \frac{1}{2} \right) \Omega(r), \end{aligned}$$

where in the final equality we have used $\frac{1}{\theta} = \frac{\Omega(r)}{\log \left(\frac{\Omega(r)}{2c_2(r)} \right)}$ from the definition of θ . If $\frac{\log \left(\sqrt{2}c_1(\nu r) e^{A \left(\ln \frac{1}{1-\nu} \right)^\tau} \right)}{\log \left(\frac{\Omega(r)}{2c_2(r)} \right)} \leq \frac{1}{4}$, then

$$(5.16) \quad \Omega(\nu r) < \frac{3}{4} \Omega(r).$$

Otherwise, $\frac{\log \left(\sqrt{2}c_1(\nu r) e^{A \left(\ln \frac{1}{1-\nu} \right)^\tau} \right)}{\log \left(\frac{\Omega(r)}{2c_2(r)} \right)} > \frac{1}{4}$, which can be rewritten as

$$(5.17) \quad \Omega(\nu r) \leq \Omega(r) < 2c_2(r) 4c_1(\nu r)^4 e^{4A \left(\ln \frac{1}{1-\nu} \right)^\tau}.$$

Altogether, we have shown that if (5.15) holds, then either (5.16) or (5.17) is satisfied. This leads to

$$(5.18) \quad \Omega(\nu r) \leq \frac{3}{4} \Omega(r) + 8c_2(r)c_1(\nu r)^4 e^{4A \left(\ln \frac{1}{1-\nu} \right)^\tau}, \quad \nu_0 \leq \nu \leq 1.$$

We now set

$$\nu_j = \nu_0 + \frac{1}{\alpha} \sum_{k=0}^{j-1} (1 - \nu_0) \left(1 + \frac{j \ln(4/3)}{8A \left(\ln \frac{1}{1-\nu_0} \right)^\tau} \right)^{1/\tau}, \quad j = 1, 2, \dots,$$

where

$$\alpha = \alpha(\nu_0, A, \tau) = \sum_{k=0}^{\infty} (1 - \nu_0) \left(1 + \frac{j \ln(4/3)}{8A \left(\ln \frac{1}{1-\nu_0} \right)^\tau} \right)^{1/\tau} - 1.$$

Then

$$\nu_{j+1} - \nu_j = \frac{1}{\alpha} (1 - \nu_0) \left(1 + \frac{j \ln(4/3)}{8A \left(\ln \frac{1}{1-\nu_0} \right)^\tau} \right)^{1/\tau}$$

and

$$\frac{\nu_j}{\nu_{j+1}} \geq \frac{\nu_0}{\nu_1} = \frac{\nu_0}{\nu_0 + \frac{(1-\nu_0)}{\alpha} \left(1 + \frac{\ln(4/3)}{8A \left(\ln \frac{1}{1-\nu_0} \right)^\tau} \right)^{1/\tau}} > \nu_0,$$

by the definition of α . Thus, $\nu_0 < \nu_j/\nu_{j+1} < 1$ for all $j \geq 0$, and

$$0 < \nu_0 < \nu_1 < \dots < \nu_j < \dots < 1.$$

Also note that ν_j is chosen such that

$$e^{4A\left(\ln \frac{1}{\nu_{j+1}-\nu_j}\right)^\tau} = \left(e^{4A\left(\ln \frac{1}{1-\nu_0}\right)^\tau} \cdot \left(\frac{4}{3}\right)^{j/2} \right)^{\left(1 + \frac{\ln \alpha}{\ln \frac{1}{1-\nu_0} \left(1 + \frac{j \ln(4/3)}{8A\left(\ln \frac{1}{1-\nu_0}\right)^\tau}\right)^{1/\tau}}\right)^\tau}$$

Then, for $j \geq 0$, from (5.18) we have

$$\begin{aligned} \Omega(\nu_j r) &= \Omega\left(\frac{\nu_j}{\nu_{j+1}} \nu_{j+1} r\right) < \frac{3}{4} \Omega(\nu_{j+1} r) + 2c_2(\nu_{j+1} r) c_1 \left(\frac{\nu_j}{\nu_{j+1}} \nu_{j+1} r\right)^4 e^{4A\left(\ln \frac{1}{1-\frac{\nu_j}{\nu_{j+1}}}\right)^\tau} \\ &\leq \frac{3}{4} \Omega(\nu_{j+1} r) + 2c_2(\nu_{j+1} r) c_1 (\nu_j r)^4 e^{4A\left(\ln \frac{1}{\nu_{j+1}-\nu_j}\right)^\tau} \\ &\leq \frac{3}{4} \Omega(\nu_{j+1} r) + 2c_2(\nu_{j+1} r) c_1 (\nu_j r)^4 \left(e^{4A\left(\ln \frac{1}{1-\nu_0}\right)^\tau} \cdot \left(\frac{4}{3}\right)^{j/2} \right)^{\left(1 + \frac{\ln \alpha}{\ln \frac{1}{1-\nu_0} \left(1 + \frac{j \ln(4/3)}{8A\left(\ln \frac{1}{1-\nu_0}\right)^\tau}\right)^{1/\tau}}\right)^\tau} \\ &\leq \frac{3}{4} \Omega(\nu_{j+1} r) + 2c_2(\nu_{j+1} r) c_1 (\nu_j r)^4 e^{4A\left(\ln \frac{1}{1-\nu_0}\right)^\tau} \cdot C \cdot \left(\frac{4}{3}\right)^{\frac{j}{2} + Cj^{1-1/\tau}} \end{aligned}$$

Concatenating these inequalities, we obtain

$$\begin{aligned} \Omega(\nu_0 r) &< \left(\frac{3}{4}\right)^j \Omega(\nu_j r) + \sum_{k=0}^{j-1} \left(\frac{3}{4}\right)^k 2c_2(\nu_{k+1} r) c_1 (\nu_k r)^4 e^{4A\left(\ln \frac{1}{1-\nu_0}\right)^\tau} \cdot C \cdot \left(\frac{4}{3}\right)^{\frac{k}{2} + Ck^{1-1/\tau}} \\ &< \left(\frac{3}{4}\right)^j \Omega(r) + 2c_2^*(r) c_1^*(r)^4 e^{4A\left(\ln \frac{1}{1-\nu_0}\right)^\tau} \cdot C \sum_{k=0}^{j-1} \left(\frac{3}{4}\right)^{\frac{k}{2} - Ck^{1-1/\tau}}, \end{aligned}$$

where c_1^* and c_2^* are given by (5.11) and where we have used both that $\nu_k \leq 1$ and, since Ω is increasing, that $\Omega(\nu_j r) \leq \Omega(r)$. Letting $j \rightarrow \infty$, we obtain

$$\Omega(\nu r) < C(\nu_0, A, \tau) c_2^*(r) c_1^*(r)^4.$$

Thus we have

$$(5.19) \quad \operatorname{esssup}_{x \in B(y, \nu_0 r)} w(x) < \exp\left(C(\nu_0, A, \tau) c_2^*(r) c_1^*(r)^4\right) e^a$$

in the case that (5.15) is satisfied.

Finally, in the case that (5.15) is violated, then either $\frac{1}{\Omega(r)} \log\left(\frac{\Omega(r)}{2c_2(r)}\right) \leq 0$, in which case

$$\Omega(\nu r) < \Omega(r) \leq 2c_2(r),$$

or $\frac{1}{\Omega(r)} \log\left(\frac{\Omega(r)}{2c_2(r)}\right) \geq 1/2$, which implies that

$$(5.20a) \quad 0 < \frac{\Omega(r)}{2} - \log \Omega(r) < \log\left(\frac{1}{2c_2(r)}\right).$$

However, $c_2 \geq 1$ implies $\log(1/c_2) < 0$, and so (5.20a) is not possible. Thus, in the case that (5.15) is violated, we have the inequality

$$\operatorname{esssup}_{y \in B(x, \nu r)} w(y) \leq \exp(2c_2(r)) e^a.$$

Lemma 49 now follows from this last inequality and (5.19) by taking

$$b = \max \left\{ \exp \left[C(\nu_0, A, \tau) c_2^*(r) c_1^*(r)^4 \right], \exp[2c_2(r)] \right\} = \exp \left[C(\nu_0, A, \tau) c_2^*(r) c_1^*(r)^4 \right].$$

■

1.4. The straight across Poincaré estimate. In order to obtain Theorem 51 below, we want to apply Lemma 49 to the reciprocal $\frac{1}{u}$ of a positive subsolution \bar{u} . The following lemma shows that (5.9) actually holds for *both* \bar{u} and $\frac{1}{\bar{u}}$. Recall that the doubling increment $\delta_y(r)$ is defined so that

$$(5.21) \quad |B(y, r - \delta_y(r))| = \frac{1}{2} |B(y, r)|.$$

We will also use the following specific cutoff Lipschitz function ϕ_r satisfying

$$(5.22) \quad \begin{cases} \operatorname{supp}(\phi_r) & \subseteq B(y, r + \delta_y(r)) \\ \{x : \phi_r(x) = 1\} & \supseteq B(y, r + \delta_y(r)/2) \\ \|\nabla_A \phi_r\|_{L^\infty(B(y, r + \delta_y(r)))} & \leq \frac{C}{\delta_y(r)}. \end{cases}$$

LEMMA 50. *Let $u \in W_A^{1,2}(\Omega)$ be a nonnegative weak solution of (3.1) in Ω , let $B = B(y, r) \subset B(y, r + \delta_y(r)) \subset \Omega$, and let $\bar{u} = u + m(r)$, $m(r) = r^2 \|\phi\|_{L^\infty}$ or more generally $\|\phi\|_{X(B(y, r))}$ with $r \leq r_0$. Assume that $\delta_y(r)$ satisfies $\delta_y(r) \leq r$ for all $y \in \Omega$ and $0 < r < \operatorname{dist}(y, \partial\Omega)$. Then there exists a constant C_W depending on the constant in the Poincaré inequality such that for all $s > 0$*

$$(5.23) \quad s |\{x \in B : \log \bar{u} > s + \langle \log \bar{u} \rangle_B\}| < C_W \frac{|B| r}{\delta_y(r)}, \quad \text{and}$$

$$(5.24) \quad s \left| \left\{ x \in B : \log\left(\frac{1}{\bar{u}}\right) > s - \langle \log \bar{u} \rangle_B \right\} \right| < C_W \frac{|B| r}{\delta_y(r)},$$

where $B = B(y, r) \subset B(y, r + \delta_y(r)) \subset \Omega$.

PROOF. As before, we set $\bar{u} = u + r^2 \|\phi\|_{L^\infty}$ or more generally $\bar{u} = u + \|\phi\|_{X(B(y, r))}$ if $\phi \not\equiv 0$, and $\bar{u} = u + m$ for $m > 0$. In the latter case we will let $m \rightarrow 0$ at the end. It is easy to check that $\log \bar{u} \in W_A^{1,2}(\Omega)$, and for any $s > 0$ we have

$$s |\{x \in B : \log \bar{u} - \langle \log \bar{u} \rangle_B > s\}| < \int_B |\log \bar{u} - \langle \log \bar{u} \rangle_B|.$$

Applying the (1, 1) Poincaré inequality (1.23) we obtain

$$s |\{x \in B : \log \bar{u} - \langle \log \bar{u} \rangle_B > s\}| \leq \int_B |\log \bar{u} - \langle \log \bar{u} \rangle_B| \leq C_P r \int_B |\nabla_A \log \bar{u}|.$$

Therefore, in order to prove (5.23) it is enough to show

$$(5.25) \quad \int_B |\nabla_A \log \bar{u}| \leq C \frac{|B|}{\delta_y(r)}.$$

Consider equation (3.2) and substitute $w = \frac{\varphi^2}{\bar{u}}$ with $\varphi \in W_0^{1,2}(B(y, r + \delta_y(r)))$ as in (5.22) to obtain

$$\begin{aligned}
 & \int_{B(y, r + \delta_y(r))} \varphi^2 |\nabla_A \log \bar{u}|^2 \\
 &= \int_{B(y, r + \delta_y(r))} \phi \frac{\varphi^2}{\bar{u}} + 2 \int_{B(y, r + \delta_y(r))} \varphi (\nabla \log \bar{u})^T A \nabla \varphi \\
 (5.26) \quad & \leq \frac{1}{r^2} |B(y, r + \delta_y(r))| + \frac{C}{\delta_y(r)} \int_{B(y, r + \delta_y(r))} \varphi |\nabla_A \log \bar{u}|,
 \end{aligned}$$

where we have used Hölder's inequality and the third property in (5.22).

Now, by the Hölder and Cauchy-Schwarz inequalities we have

$$2 \int_{B(y, r + \delta_y(r))} \varphi |\nabla_A \log \bar{u}| \leq \varepsilon |B(y, r + \delta_y(r))| + \varepsilon^{-1} \int_{B(y, r + \delta_y(r))} \varphi^2 |\nabla_A \log \bar{u}|^2$$

for all $\varepsilon > 0$. Taking $\varepsilon = \frac{C}{\delta_y(r)}$ and multiplying by $\frac{C}{\delta_y(r)}$ it follows that

$$\frac{2C}{\delta_y(r)} \int_{B(y, r + \delta_y(r))} \varphi |\nabla_A \log \bar{u}| - \left(\frac{C}{\delta_y(r)} \right)^2 |B(y, r + \delta_y(r))| \leq \int_{B(y, r + \delta_y(r))} \varphi^2 |\nabla_A \log \bar{u}|^2.$$

Applying this estimate on the left of (5.26) we obtain

$$\begin{aligned}
 & \frac{C}{\delta_y(r)} \int_{B(y, r + \delta_y(r))} \varphi |\nabla_A \log \bar{u}| - \left(\frac{C}{\delta_y(r)} \right)^2 |B(y, r + \delta_y(r))| \\
 & \leq \frac{1}{r^2} |B(y, r + \delta_y(r))| + \frac{C}{\delta_y(r)} \int_{B(y, r + \delta_y(r))} \varphi |\nabla_A \log \bar{u}|,
 \end{aligned}$$

and, re-arranging terms,

$$\begin{aligned}
 2 \int_{B(y, r + \delta_y(r))} \varphi |\nabla_A \log \bar{u}| & \leq \left(\frac{\delta_y(r)}{Cr^2} + \frac{C}{\delta_y(r)} \right) |B(y, r + \delta_y(r))| \\
 & \leq \frac{C}{\delta_y(r)} |B(y, r + \delta_y(r))| \\
 & \leq \frac{C}{\delta_y(r)} |B(y, r)|,
 \end{aligned}$$

where in the second inequality we used that $\delta_y(r)/r \leq 1$ and that $C \geq 1$, and in the last inequality the definition of the duplicating rate δ_y , (5.21). This concludes the proof of (5.25) and so (5.23) is established. The proof of (5.24) proceeds in a similar way:

$$\begin{aligned}
 s |\{x \in B : \log(1/\bar{u}) > s - \langle \log \bar{u} \rangle_B\}| &= |\{x \in B : \log(1/\bar{u}) - \langle \log(1/\bar{u}) \rangle_B > s\}| \\
 &\leq \int_B |\log(1/\bar{u}) - \langle \log(1/\bar{u}) \rangle_B| \\
 &\leq C_p r \int_B |\nabla_A \log(1/\bar{u})|.
 \end{aligned}$$

Then (5.24) follows from (5.25) after noting that $|\nabla_A \log(1/\bar{u})| = |\nabla_A \log(\bar{u})|$. In the case $\phi \equiv 0$ we note that the constants are independent of $m > 0$, so the result follows for $\bar{u} = u$ by letting $m \rightarrow 0$. ■

1.5. The infimum half of the Harnack inequality. We can now establish half of a weak version of the Harnack inequality.

THEOREM 51. *Assume that $\varphi(r)$ and $\Phi(t) = \Phi_m(t)$ with $m > 2$ satisfy the Sobolev bump inequality (1.5), that the $(1, 1)$ Poincaré inequality (1.23) holds, and that a nonstandard sequence of Lipschitz cutoff functions exists. Let u be a nonnegative weak solution of $\mathcal{L}u = \phi$ in $B(y, r)$ with A -admissible ϕ . Then, for any $0 < \nu_0 \leq \nu < 1$ as in Lemma 49, the weak solution u satisfies the following half Harnack inequality,*

$$(5.27) \quad \frac{1}{b} e^{\langle \log(u + \|\phi\|_{X(B(y, r))}) \rangle_B} \leq \operatorname{essinf}_{x \in B(y, \nu r)} \left(u(x) + \|\phi\|_{X(B(y, r))} \right),$$

where with c_j^* as in (5.11),

$$b^2 = \exp \left(\frac{64c_1^*(r)^4 c_2^*(r)}{C(1-\nu)^{4\tau}} \right).$$

PROOF. By the Inner Ball inequality in Theorem 48, for $\bar{u} = u + \|\phi\|_{X(B(y, r))}$ and $\beta \in (-1/2, 0)$, there exist a locally bounded function $c_1(y, r)$ and a constant τ such that for all $\nu_0 \leq \nu < 1$

$$(5.28) \quad \operatorname{esssup}_{x \in B(y, \nu r)} \bar{u}(x)^\beta \leq c_1(y, \nu r) e^{A(\ln \frac{1}{1-\nu})^\tau} \left\{ \frac{1}{|B|} \int_B \bar{u}^{2\beta} \right\}^{1/2}.$$

Indeed, the constant $C(\varphi, m, r) = e^{C(\ln K)^m}$ can be written in the form

$$C(\varphi, m, r) = e^{C(\ln \frac{\varphi(r)}{\delta(r)})^m} e^{C(\ln \frac{1}{1-\nu})^m}$$

using the definition of K in (5.2). Also, by Lemma 50 we have that there exists C_W such that for all $s > 0$

$$s |\{x \in B : \log(1/\bar{u}) > s - \langle \log \bar{u} \rangle_B\}| < C_W \frac{|B| r}{\delta_y(r)}.$$

Then, using (5.28) for the range $-1/2 < \beta < 0$, we apply Lemma 49 to $w = \frac{1}{\bar{u}}$ with $c_2(y, r) = \frac{C_W r}{\delta_y(r)}$ and $a = -\langle \log \bar{u} \rangle_B$ to obtain

$$(5.29) \quad \operatorname{esssup}_{B(y, \nu r)} \left(\frac{1}{\bar{u}} \right) \leq b e^{-\langle \log \bar{u} \rangle_B}.$$

■

2. Bombieri and half Harnack for a solution $u < \frac{1}{M}$

As a result of the considerations in Section 5, we must abandon one of the three numbered properties listed there, and it will be the submultiplicativity. Our new bump function will instead be supermultiplicative, and this has already played an important role in the proof in our Orlicz-Sobolev bump inequality.

For $t \leq \frac{1}{M}$ for some fixed M large enough, and m odd recall that we defined

$$\Psi(t) = A e^{((\ln t)^{\frac{1}{m}} - 1)^m},$$

where $A > 0$ was chosen so that $\Psi((0, \frac{1}{M})) = (0, \frac{1}{M})$, namely,

$$A = e^{((\ln M)^{1/m} + 1)^m - \ln M} > 1.$$

More generally, for $m > 1$ we define

$$\begin{aligned}\Psi(t) &= Ae^{-\left(\ln \frac{1}{t}\right)^{\frac{1}{m}+1}}; \\ A &= e^{((\ln M)^{1/m}+1)^m - \ln M} > 1.\end{aligned}$$

The inverse function Ψ^{-1} is given by

$$\begin{aligned}s &= \Psi(t) = Ae^{-\left(\ln \frac{1}{t}\right)^{\frac{1}{m}+1}}; \\ \left(\left(\ln \frac{1}{t}\right)^{\frac{1}{m}} + 1\right)^m &= \ln \frac{A}{s}; \\ \ln \frac{1}{t} &= \left(\left(\ln \frac{A}{s}\right)^{\frac{1}{m}} - 1\right)^m; \\ \Psi^{-1}(s) &= t = e^{-\left(\left(\ln \frac{A}{s}\right)^{\frac{1}{m}} - 1\right)^m}.\end{aligned}$$

We will below modify the graph of Ψ for $t \geq \frac{1}{M}$ to be linear.

But first note that the following properties hold M sufficiently large.

(1) For $t \leq 1/M$,

$$(5.30) \quad \Psi^{-1}(t) = \begin{cases} e^{\left(\ln \frac{1}{A}\right)^{\frac{1}{m}+1}} & \text{if } m \text{ is odd} \\ e^{-\left(\ln \frac{1}{A}\right)^{\frac{1}{m}-1}} & \text{if } m > 1 \end{cases},$$

(2) $\Psi(t)$ is increasing on $(0, \frac{1}{M})$,

(3) $\Psi(t)$ is convex,

(4) $\Psi(t)$ is A -supermultiplicative, i.e. $A\Psi(ab) \geq \Psi(a)\Psi(b)$.

For large values of t , $t \geq \frac{1}{M}$ we now extend $\Psi(t)$ to be a linear function with the same slope

$$\Psi'\left(\frac{1}{M}\right) \text{ at } t = \frac{1}{M}. \text{ We then have } \Psi^{-1}(t) = \begin{cases} e^{\left(\ln \frac{1}{A}\right)^{\frac{1}{m}+1}} & \text{if } m \text{ is odd} \\ e^{-\left(\ln \frac{1}{A}\right)^{\frac{1}{m}-1}} & \text{if } m > 1 \end{cases} \text{ for } t \leq \frac{1}{M}, \text{ and}$$

that Ψ^{-1} is linear for $t \geq \frac{1}{M}$. The following properties of $\Psi^{-1}(t)$ follow from the properties of $\Psi(t)$ above and the linearity of the extension.

(1) $\Psi^{-1}\left((0, \frac{1}{M})\right) = (0, \frac{1}{M})$,

(2) $\Psi^{-1}(t)$ is increasing,

(3) $\Psi^{-1}(t)$ is concave.

2.1. Iteration and the Inner Ball inequality for super solutions $\Psi^{(-N)}u$ with $N \geq 1$.

Recall that $\Psi(t) = A_m e^{-\left(\ln \frac{1}{t}\right)^{\frac{1}{m}+1}}$ for $t > 0$ small and some $m > 2$. Assume we have the Orlicz-Sobolev inequality with Ψ bump and superradius φ :

$$(5.31) \quad \Psi^{(-1)}\left(\int_B \Psi(w)\right) \leq C\varphi(r(B)) \int_B |\nabla_A(w)| d\mu, \quad w \in Lip_{\text{compact}}(B).$$

We will iterate the following Moser inequality to obtain the Inner Ball inequality. Recall the sequence of balls $\{B_n\}_{n=0}^\infty$ and cutoff functions ψ_n defined in (1.22).

LEMMA 52. Let $u < \frac{1}{M}$ be a weak supersolution to $Lu = \phi$ with ϕ admissible and for $N \geq 1$, let $h_N(t) = \sqrt{\Psi^{(-N)}(t)}$ be as in (3.14). Define $\Theta(t) = \left(\ln \frac{1}{\Psi(t)}\right)^{\frac{1}{m}} t$ as in (3.16). Then Θ is concave and for all $n \geq 0$,

$$(5.32) \quad \int_{B_{n+1}} h_{N-1}(u)^2 d\mu \leq \Psi \left(C_n \Theta \left(\int_{B_n} h_N(u)^2 d\mu \right) \right), \quad N \geq 1.$$

Similarly, if $u < \frac{1}{M}$ is a weak subsolution to $Lu = \phi$ with ϕ admissible, and if $h_N(t) = \sqrt{\Psi^{(N)}(t)}$, then

$$\int_{B_{n+1}} h_{N+1}(u)^2 d\mu \leq \Psi \left(C_n \Theta \left(\int_{B_n} h_N(u)^2 d\mu \right) \right), \quad N \geq 1.$$

COROLLARY 53. Let $h_k(t) = \sqrt{\Psi^{(k)}(t)}$ for $k \in \mathbb{Z}$. Then we have for all $n \geq 0$,

$$(5.33) \quad \begin{aligned} \int_{B_{n+1}} \Psi^{(k+1)}(u) d\mu &\leq \Psi \left(C_n \Theta \left(\int_{B_n} \Psi^{(k)}(u) d\mu \right) \right), \quad k \in \mathbb{Z}, \\ \int_{B_{n+1}} h_{k+1}(u)^2 d\mu &\leq \Psi \left(C_n \Theta \left(\int_{B_n} h_k(u)^2 d\mu \right) \right), \quad k \in \mathbb{Z}. \end{aligned}$$

PROOF OF LEMMA 52. We apply the inhomogeneous Orlicz-Sobolev inequality (5.31) with $w = \psi_n^2 h(u)^2$ to obtain

$$\Psi^{(-1)} \left(\int_{B_{n+1}} \Psi(h(u)^2) d\mu \right) \leq C\varphi(r_n) \int_{B_n} \|\nabla_A(\psi_n^2 h(u)^2)\| d\mu.$$

For the right hand side we define $k'(t) = \sqrt{|\Lambda(t)|}$ to write

$$\begin{aligned} \int \|\nabla_A(\psi_n^2 h(u)^2)\| &\leq 2 \int \psi_n^2 h(u) h'(u) \|\nabla_A u\| + 2 \int \psi_n \|\nabla_A \psi_n\| h(u)^2 \\ &= 2 \int \psi_n \frac{h(u) h'(u)}{\sqrt{|\Lambda(u)|}} \psi_n k'(u) \|\nabla_A u\| + 2 \sqrt{\int \psi_n^2 h(u)^2} \sqrt{\int \|\nabla_A \psi_n\|^2 h(u)^2} \\ &\leq 2 \left(\int \psi_n^2 \frac{(h'(u))^2}{|\Lambda(t)|} h(u)^2 d\mu \right)^{\frac{1}{2}} \left(\int \psi_n^2 \|\nabla_A k(u)\|^2 d\mu \right)^{\frac{1}{2}} \\ &\quad + 2 \left(\int \psi_n^2 h(u)^2 \right)^{\frac{1}{2}} \left(\int \|\nabla_A \psi_n\|^2 h(u)^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Now we use Cauchy-Schwartz to dominate the above by

$$\begin{aligned} &\frac{1}{\varphi(r_n)} \int \psi_n^2 \frac{(h'(u))^2}{|\Lambda(t)|} h(u)^2 d\mu + \varphi(r_n) \int \psi_n^2 \|\nabla_A k(u)\|^2 d\mu \\ &\quad + \frac{1}{\varphi(r_n)} \int \psi_n^2 h(u)^2 + \varphi(r_n) \int \|\nabla_A \psi_n\|^2 h(u)^2. \end{aligned}$$

Combining these three inequalities we obtain

$$\begin{aligned} \Psi^{-1} \left(\int_{B_{n+1}} \Psi \left(h(u)^2 \right) d\mu \right) &\leq C \left\{ \int \psi_n^2 \frac{(h'(u))^2}{|\Lambda(t)|} h(u)^2 d\mu + \int \psi_n^2 h(u)^2 \right\} \\ &\quad + C \varphi(r_n)^2 \left\{ \int \psi_n^2 \|\nabla_A k(u)\|^2 d\mu + \int \|\nabla_A \psi_n\|^2 h(u)^2 \right\}. \end{aligned}$$

Suppose first that $u < \frac{1}{M}$ is a weak supersolution to $Lu = \phi$ with ϕ admissible, and that $h_N(t) = \sqrt{\Psi^{(-N)}(t)}$. Now recall from Lemma 36 that

$$\int \psi_n^2 \|\nabla_A k_N(u)\|^2 d\mu \leq C \int \left(\psi_n^2 + \|\nabla_A \psi_n\|^2 \right) \frac{(h'_N(u))^2}{|\Lambda_N(t)|} h_N(u)^2 d\mu,$$

and so using $\frac{(h'_N(u))^2}{|\Lambda_N(t)|} \geq c > 0$ we have altogether that

$$\begin{aligned} \Psi^{-1} \left(\int_{B_{n+1}} \Psi \left(h_N(u)^2 \right) d\mu \right) &\leq C \int \left(\psi_n^2 + \varphi(r_n)^2 \|\nabla_A \psi_n\|^2 \right) \frac{(h'_N(u))^2}{|\Lambda_N(t)|} h_N(u)^2 d\mu \\ &\leq C(\varphi, n, r_0)^2 \int_{B_n} \frac{(h'_N(u))^2}{|\Lambda_N(t)|} h_N(u)^2 d\mu, \end{aligned}$$

where

$$C(\varphi, n, r_0) \equiv \sqrt{C} (1 + \varphi(r_n) \|\nabla_A \psi_n\|_\infty).$$

Now we use the second inequality in Lemma 39 to conclude that

$$\Psi^{-1} \left(\int_{B_{n+1}} \Psi \left(h_N(u)^2 \right) d\mu \right) \leq \frac{C(\varphi, n, r_0)^2}{m - \frac{3}{2}} \Theta \left(\int_{B_n} h_N(u)^2 d\mu \right) = C_n \Theta \left(\int_{B_n} h_N(u)^2 d\mu \right),$$

where

$$C_n = \frac{C(\varphi, n, r_0)^2}{m - \frac{3}{2}} = \frac{1}{m - \frac{3}{2}} C (1 + \varphi(r_n) \|\nabla_A \psi_n\|_\infty)^2.$$

The case when $u < \frac{1}{M}$ is a weak subsolution to $Lu = \phi$ with ϕ admissible, and when $h_N(t) = \sqrt{\Psi^{(N)}(t)}$, is handled in similar fashion. ■

Now we define a sequence $\{B_n\}_{n=0}^\infty$ of positive numbers by

$$(5.34) \quad B_0 = \int_{B(0, r_0)} \Psi^{(-N)}(u) d\mu_{r_0}, \quad B_{n+1} = \Psi(C_n \Theta(B_n)),$$

where $\Theta(t) = t \left(\ln \frac{1}{\Psi(t)} \right)^{\frac{1}{m}} = t^{1 - \frac{1}{m} \frac{\ln \ln \frac{1}{\Psi(t)}}{\ln \frac{1}{t}}}$. Here B_n refers to a positive number rather than a ball, but the meaning should be clear from the context.

REMARK 54. Using $\Psi(t) = Ae^{-\left(\left(\ln \frac{1}{t}\right)^{\frac{1}{m}} + 1\right)^m}$ we obtain

$$\ln \ln \frac{1}{\Psi(t)} = \ln \left[\left(\left(\ln \frac{1}{t} \right)^{\frac{1}{m}} + 1 \right)^m - \ln A \right],$$

and hence that

$$\begin{aligned}\Theta(t) &= t \left(\ln \frac{1}{\Psi(t)} \right)^{\frac{1}{m}} = t^{1 - \frac{1}{m} \frac{\ln \ln \frac{1}{\Psi(t)}}{\ln \frac{1}{t}}} = t^{1 - \frac{1}{m} \frac{\ln \left[\left(\ln \frac{1}{t} \right)^{\frac{1}{m} + 1} \right]^{m - \ln A}}{\ln \frac{1}{t}}} \\ &\approx t^{1 - \frac{1}{m} \frac{\ln \ln \frac{1}{t}}{\ln \frac{1}{t}}} = e^{-(\ln \frac{1}{t})(1 - \frac{1}{m} \ln \ln \frac{1}{t})} = e^{-\ln \frac{1}{t} + \frac{1}{m} \ln \ln \frac{1}{t}},\end{aligned}$$

upon using the approximation

$$\ln \ln \frac{1}{\Psi(t)} \approx \ln \ln \frac{1}{t}.$$

We also have

$$(\Theta \circ \Psi)(t) = \Psi(t) \left(\ln \frac{1}{\Psi^{(2)}(t)} \right)^{\frac{1}{m}},$$

and since

$$\ln \frac{1}{\Psi^{(2)}(t)} = -\ln A + \left(\left(\ln \frac{1}{t} \right)^{\frac{1}{m}} + 2 \right)^m \approx \ln \frac{1}{t},$$

we have the approximation

$$(\Theta \circ \Psi)(t) \approx \Psi(t) \left(\ln \frac{1}{t} \right)^{\frac{1}{m}}.$$

Note that since $(\Theta \circ \Psi)(t) \gg \Psi(t)$ as $t \rightarrow 0$, it will be harder for iterates of $\Psi(C_n \Theta(\cdot))$ to ‘catch up’ with iterates of Ψ .

The inequality (5.33) and a basic induction using

$$\begin{aligned}B_{n+1} &= \Psi(C_n \Theta(B_n)) \\ &= \Psi(C_n \Theta(\Psi(C_{n-1} \Theta(B_{n-1})))) \\ &= \Psi(C_n \Theta(\Psi(C_{n-1} \Theta(\Psi(C_{n-2} \Theta(B_{n-2})))))) \\ &= \Psi(C_n (\Theta \circ \Psi)(C_{n-1} (\Theta \circ \Psi)(C_{n-2} \Theta(B_{n-2})))) \\ &= \Psi(C_n \Theta \circ \Psi(C_{n-1} \Theta \circ \Psi(C_{n-2} \Theta(B_{n-2})))) \\ &\vdots \\ &= \Psi(C_n \Theta \circ \Psi(C_{n-1} \Theta \circ \Psi(C_{n-2} \dots \Theta \circ \Psi(C_{n-\ell} \Theta(B_{n-\ell}))) \dots))\end{aligned}$$

shows that

$$(5.35) \quad \int_{B(0, r_n)} \Psi^{(n-N)}(u) d\mu_{r_n} \leq B_n.$$

At this point we require the following two properties of the function Ψ relative to the solution u :

$$(5.36) \quad \liminf_{n \rightarrow \infty} \left[\Psi^{(n)} \right]^{-1} \left(\int_{B(0, r_n)} \Psi^{(n-N)}(u) d\mu_{r_n} \right) \geq \|u\|_{L^\infty(\mu_{r_\infty})}^2,$$

and

$$(5.37) \quad \liminf_{n \rightarrow \infty} \left[\Psi^{(n)} \right]^{-1}(B_n) \leq C(\varphi, m, r)$$

The combination of (5.36), (5.35) and (5.37) in sequence immediately finishes the proof:

$$(5.38) \quad \begin{aligned} \left\| \Psi^{(-N)}(u) \right\|_{L^\infty(\mu_{r_\infty})} &\leq \liminf_{n \rightarrow \infty} \left[\Psi^{(n)} \right]^{-1} \left(\int_{B(0, r_n)} \Psi^{(n-N)}(u) d\mu_{r_n} \right) \\ &\leq \liminf_{n \rightarrow \infty} \left[\Psi^{(n)} \right]^{-1} (B_n) \leq C(\varphi, m, r). \end{aligned}$$

The two properties (5.36) and (5.37) now follow from Lemmas 56 and 58 below with

$$B_0 = \int_{B(0, r_0)} \Psi^{(-N)}(u) d\mu_{r_0} \leq B'_0(m, K) = e^{-C(\ln K)^m}.$$

We now give an estimate for the constant $C(\varphi, m, r)$. Recall $C(\varphi, m, r) = C^*$ and by (5.44),

$$C^* \equiv A_0 = e^{-a_0^m} \leq e^{-(1 + (\ln M)^{1/m} + \gamma + \ln K)^m},$$

i.e.

$$(5.39) \quad C(\varphi, m, r) = e^{-a_0^m} \lesssim \exp \left[- \left(1 + (\ln M)^{1/m} + \gamma + \ln K \right)^m \right].$$

Recall that $C_n = \frac{1}{m-\frac{3}{2}} C (1 + \varphi(r_n) \|\nabla_A \psi_n\|_\infty)^2$. Using (1.22), we can dominate C_n by

$$(5.40) \quad C_n \leq C (1 + \varphi(r_n) \|\nabla_A \psi_n\|_\infty)^2 \leq C \left(1 + G \frac{\varphi(r_n)}{(1-\nu)\delta(r_n)} n^2 \right)^2 \leq K n^\gamma,$$

where the constant K can be estimated as follows

$$K = \left(\frac{C\varphi(r_0)}{(1-\nu)\delta(r_0)} \right)^2.$$

Just as in the previous section, an argument based on linearity and tracking constants using (5.38) gives this.

THEOREM 55. *Assume that $\varphi(r)$ and $\Phi(t) = \Psi_m(t)$ with $m > 2$ satisfy the Sobolev bump inequality (1.5), and that a nonstandard sequence of Lipschitz cutoff functions exists. Let u be a nonnegative bounded weak solution to the equation $\mathcal{L}u = \phi$ in $B(0, r)$, so that $\|\phi\|_{X(B(0, r))} < \infty$. Then we have a constant $C(\varphi, m, r)$, such that*

$$\left\| \Psi^{(-N)}(u + \|\phi\|_{X(B(0, r))}) \right\|_{L^\infty(B(0, r/2))} \leq C(\varphi, m, r) \left\| \Psi^{(-N)}(u + \|\phi\|_{X(B(0, r))}) \right\|_{L^1(d\mu_r)},$$

where

$$C(\varphi, m, r) \leq \exp [C'(m) (C + \ln K)^m].$$

2.1.1. The recursion lemmas. We now introduce some further notation. Recall $C_n \leq K n^\gamma$ by (5.40). Given $B_0 > 0$ and $A_0 > 0$ we define two sequences (with the B_n now larger than in (5.34))

$$B_n \equiv \Psi(K n^\gamma \Theta(B_{n-1})),$$

and

$$A_n = e^{((\ln(A_{n-1}))^{\frac{1}{m}} - 1)^m},$$

First note that provided $B_n \leq 1/M$ we have

$$B_n = A e^{-\left((-\ln(K n^\gamma \Theta(B_{n-1})))^{\frac{1}{m}} + 1 \right)^m}$$

provided $Kn^\gamma \Theta(B_{n-1}) < 1/M$. Moreover, since $A \geq 1$ we have $A_n \leq \Psi^{(n)}(A_0)$. Next,

$$A_n = e^{\left((\ln(A_0))^{\frac{1}{m}} - n\right)^m}$$

and denoting $a_n \equiv -(\ln A_n)^{\frac{1}{m}}$ we have

$$a_n = a_{n-1} + 1 = a_0 + n.$$

Also with $b_n \equiv -(\ln \frac{B_n}{A})^{\frac{1}{m}} > 0$, we have from the above that

$$b_n = (-\ln(Kn^\gamma \Theta(B_{n-1})))^{\frac{1}{m}} + 1.$$

We now rewrite $\Theta(B_{n-1})$ in terms of b_{n-1}

$$\begin{aligned} \Theta(B_{n-1}) &= B_{n-1} \left(\left[(-\ln B_{n-1})^{\frac{1}{m}} + 1 \right]^m - \ln A \right)^{\frac{1}{m}} \\ &= e^{-b_{n-1}^m} A \left(\left[(b_{n-1}^m - \ln A)^{\frac{1}{m}} + 1 \right]^m - \ln A \right)^{\frac{1}{m}}, \end{aligned}$$

so that we have

$$b_n = \left[b_{n-1}^m - \ln(Kn^\gamma) - \ln A - \frac{1}{m} \ln \left(\left[(b_{n-1}^m - \ln A)^{\frac{1}{m}} + 1 \right]^m - \ln A \right) \right]^{1/m} + 1,$$

provided $b_n \geq (\ln M)^{1/m}$ for all n . For convenience in notation we will replace K by KA so that the term $-\ln(Kn^\gamma) - \ln A$ is replaced by $-\ln(Kn^\gamma)$ and we have

$$(5.41) \quad b_n = \left[b_{n-1}^m - \ln(Kn^\gamma) - \frac{1}{m} \ln \left(\left[(b_{n-1}^m - \ln A)^{\frac{1}{m}} + 1 \right]^m - \ln A \right) \right]^{1/m} + 1.$$

We need the following two lemmas to obtain (5.36) and (5.37). For (5.37) we need the following comparison Lemma.

LEMMA 56. *Given any $m > 2$, $K > e$ and $\gamma > 0$, consider the sequence defined by*

$$B_0 > 0, \quad B_{n+1} = \Psi(K(n+1)^\gamma \Theta(B_n)).$$

Then for each sufficiently small positive number $B_0 < B'_0(m, K)$, there exists a positive number $C = C(B_0, m, M, K, \gamma) < 1/M$, such that the inequality

$$\Psi^{(n)}(C) \geq B_n$$

holds for each positive number n . The number $B'_0(m, M, K, \gamma)$ can be taken to be $e^{-C(\ln K)^m}$.

PROOF. We first note that it is enough to show that $B_n \leq A_n$ where A_n is defined as above with $A_0 = C$. In terms of b_n and a_n it suffices to show that for each sufficiently large $b_0 \geq b'_0(m, M, K, \gamma)$, there exists a number $a_0 > (\ln M)^{1/m}$, such that the inequality $b_n \geq a_0 + n$ holds for each positive number n . We will prove the claim by induction on n . Let

$$\begin{aligned} b_0 &\geq 1 + (\ln M)^{1/m} + \gamma + \ln K + \sum_{k=0}^{\infty} \frac{\ln K + (\gamma + 1) \ln(k+1)}{(k+1)^{m-1}}, \\ a_0 &= b_0 - \sum_{k=0}^{\infty} \frac{\ln K + (\gamma + 1) \ln(k+1)}{(k+1)^{m-1}}, \end{aligned}$$

and assume

$$(5.42) \quad b_n \geq b_0 + n - \sum_{k=0}^{n-1} \frac{\ln K + (\gamma + 1) \ln(k+1)}{(k+1)^{m-1}} \geq n + 1 + (\ln M)^{1/m} + \gamma + \ln K.$$

First, the inequality (5.42) is trivial when $n = 0$. Now we assume that the inequality (5.42) holds for a nonnegative integer n , and we will show that (5.42) holds for $n + 1$. Recall the definition $B_{n+1} = \Psi(K(n+1)^\gamma \Theta(B_n))$, which requires that we ensure the argument $K(n+1)^\gamma \Theta(B_n)$ of Ψ never escapes the interval $(0, 1/M)$ i.e.

$$K(n+1)^\gamma \Theta(B_n) < \frac{1}{M},$$

or equivalently

$$(5.43) \quad \Theta(B_n) = e^{-b_n^m} A \left(\left[(b_n^m - \ln A)^{\frac{1}{m}} + 1 \right]^m - \ln A \right)^{\frac{1}{m}} < \frac{1}{MK(n+1)^\gamma}$$

Using the second inequality in (5.42) and the definition $A \equiv e^{((\ln M)^{1/m} + 1)^m - \ln M}$ we have

$$\begin{aligned} \ln \Theta(B_n) &\leq \left((\ln M)^{1/m} + 1 \right)^m - \ln M - \left(n + 1 + (\ln M)^{1/m} + \gamma + \ln K \right)^m \\ &\quad + \ln \left(n + 2 + (\ln M)^{1/m} + \gamma + \ln K \right) \end{aligned}$$

Thus, to establish (5.43) it is enough to show

$$\begin{aligned} \left((\ln M)^{1/m} + 1 \right)^m - \ln M - \left(n + 1 + (\ln M)^{1/m} + \gamma + \ln K \right)^m + \ln \left(n + 2 + (\ln M)^{1/m} + \gamma + \ln K \right) \\ \leq -\ln M - \ln K - \gamma \ln(n+1) \end{aligned}$$

or equivalently

$$\begin{aligned} &\left(n + 1 + (\ln M)^{1/m} + \gamma + \ln K \right)^m \\ &\geq \left((\ln M)^{1/m} + 1 \right)^m + \ln K + \gamma \ln(n+1) + \ln \left(n + 2 + (\ln M)^{1/m} + \gamma + \ln K \right). \end{aligned}$$

Using the binomial expansion for the term on the left we have

$$\begin{aligned} &\left(n + 1 + (\ln M)^{1/m} + \gamma + \ln K \right)^m \\ &\geq (n + \gamma + \ln K)^m + m(n + \gamma + \ln K) \left((\ln M)^{1/m} + 1 \right)^{m-1} + \left((\ln M)^{1/m} + 1 \right)^m, \end{aligned}$$

and therefore it is enough to show

$$\begin{aligned} &(n + \gamma + \ln K)^m + m(n + \gamma + \ln K) \left((\ln M)^{1/m} + 1 \right)^{m-1} \\ &- \ln \left(n + 2 + (\ln M)^{1/m} + \gamma + \ln K \right) - \ln K - \gamma \ln(n+1) \geq 0. \end{aligned}$$

Clearly, the left hand side is an increasing function of n , $\ln K$, γ and $(\ln M)^{1/m}$, and we can assume $\ln K > 1$. It is easy to see that the above inequality holds for $n = \gamma = (\ln M)^{1/m} = 0$ and $\ln K = 1$, and this concludes the proof of (5.43).

We now return to the proof of the induction step and write using (5.41) and the inequality $(1-x)^{\frac{1}{m}} \geq 1 - \frac{3}{2} \frac{1}{m} x$ for $x > 0$ small,

$$\begin{aligned} b_{n+1} &= \left(b_n^m - \ln(K(n+1)^\gamma) - \frac{1}{m} \ln \left(\left[(b_n^m - \ln A)^{\frac{1}{m}} + 1 \right]^m - \ln A \right) \right)^{1/m} + 1 \\ &\geq b_n \left(1 - \frac{\ln K + \gamma \ln(n+1) + \ln \left[(b_n^m - \ln A)^{\frac{1}{m}} + 1 \right]}{b_n^m} \right)^{1/m} + 1 \\ &\geq b_n - \frac{3}{2} \frac{1}{m} \frac{\ln K + \gamma \ln(n+1) + \ln \left[(b_n^m - \ln A)^{\frac{1}{m}} + 1 \right]}{b_n^{m-1}} + 1 \\ &\geq b_n + 1 - \frac{3}{2} \frac{1}{m} \frac{\ln K + \gamma \ln(n+1) + \frac{4}{3} \ln b_n}{b_n^{m-1}}, \end{aligned}$$

where in the last inequality we used the rough bound $\ln \left[(b_n^m - \ln A)^{\frac{1}{m}} + 1 \right] \leq \frac{4}{3} \ln b_n$. Finally, using our induction assumption (5.42) we obtain

$$\begin{aligned} b_{n+1} &\geq b_0 + n + 1 - \sum_{k=0}^{n-1} \frac{\ln K + (\gamma+1) \ln(k+1)}{(k+1)^{m-1}} - \frac{3}{2} \frac{1}{m} \frac{\ln K + \gamma \ln(n+1) + \frac{4}{3} \ln b_n}{b_n^{m-1}} \\ &\geq b_0 + n + 1 - \sum_{k=0}^n \frac{\ln K + (\gamma+1) \ln(k+1)}{(k+1)^{m-1}} \end{aligned}$$

where for the last inequality we used a rough bound from (5.42), namely $b_n \geq n+1$, and the fact that $m > 2$, to obtain

$$\begin{aligned} \frac{3}{2} \frac{1}{m} \frac{\ln K + \gamma \ln(n+1) + \frac{4}{3} \ln b_n}{b_n^{m-1}} &\leq \frac{3}{4} \frac{\ln K + \gamma \ln(n+1) + \frac{4}{3} \ln(n+1)}{(n+1)^{m-1}} \\ &\leq \frac{\ln K + (\gamma+1) \ln(n+1)}{(n+1)^{m-1}}, \end{aligned}$$

using that $\frac{\ln x}{x^{m-1}}$ is decreasing. ■

REMARK 57. Note that we can estimate $a_0 \geq 1 + (\ln M)^{1/m} + \gamma + \ln K$ by our bound on b_0 in Lemma 56, and it follows that

$$(5.44) \quad C^* \equiv A_0 = e^{-a_0^m} \leq e^{-(1+(\ln M)^{1/m} + \gamma + \ln K)^m}$$

For property (5.36) we need the following variant of Lemma 41 proven above for a different function.

LEMMA 58. Given any $M_1 > M_2 > M$ and $\delta \in (0, 1)$, the inequality

$$\delta \Psi^{(n)} \left(\frac{1}{M_2} \right) \geq \Psi^{(n)} \left(\frac{1}{M_1} \right)$$

holds for each sufficiently large $n > N(M, M_1, M_2, \delta)$.

PROOF. Let $a_0 = (\ln M_2)^{\frac{1}{m}}$ and $b_0 = (\ln M_1)^{\frac{1}{m}} > a_0$, and

$$\begin{aligned} a_n &= - \left(\ln \Psi^{(n)} \left(\frac{1}{M_2} \right) \right)^{\frac{1}{m}} \\ b_n &= - \left(\ln \Psi^{(n)} \left(\frac{1}{M_1} \right) \right)^{\frac{1}{m}} \end{aligned}$$

which implies recursive relations

$$\begin{aligned} a_n &= ((a_{n-1} + 1)^m - \ln A)^{\frac{1}{m}} \\ b_n &= ((b_{n-1} + 1)^m - \ln A)^{\frac{1}{m}} \end{aligned}$$

Lemma will be proven if we show

$$(5.45) \quad \lim_{n \rightarrow \infty} b_n^m - a_n^m = \infty$$

We can write

$$(5.46) \quad b_n^m - a_n^m = (b_{n-1} + 1)^m - (a_{n-1} + 1)^m \geq m(a_{n-1} + 1)^{m-1} (b_{n-1} - a_{n-1})$$

First, it is easy to see that $b_n - a_n > b_{n-1} - a_{n-1} > \dots > b_0 - a_0$. Indeed, using the IVT and induction we get

$$\begin{aligned} b_n - a_n &= ((b_{n-1} + 1)^m - \ln A)^{\frac{1}{m}} - ((a_{n-1} + 1)^m - \ln A)^{\frac{1}{m}} \\ &\geq \frac{1}{\left(1 - \frac{\ln A}{(b_{n-1} + 1)^m}\right)^{\frac{m-1}{m}}} (b_{n-1} - a_{n-1}) > b_{n-1} - a_{n-1} > \dots > b_0 - a_0 \end{aligned}$$

Thus to show (5.45) by (5.46) we only need to show that $a_n \rightarrow \infty$ as $n \rightarrow \infty$. Using the IVT and induction similar to the above we have

$$a_n - a_{n-1} > a_1 - a_0 > 0$$

where the last inequality is due to the fact that Ψ is convex, onto, and $a_0 > M$. This gives $a_n \geq (a_1 - a_0)n \rightarrow \infty$ as $n \rightarrow \infty$ which concludes the proof. ■

2.2. Bombieri's lemma for super solutions $\Psi^{(-N)}u$ with $N \geq 1$. Now we can state and prove our Bombieri lemma for use in the concave region.

LEMMA 59. Let $0 < w < \frac{1}{M}$ be a positive, bounded and measurable function defined in a neighborhood of $B_0 = B(y_0, r_0)$. Suppose there exist positive constants τ , A , and $\nu_0 < 1$, $a \leq \ln \frac{1}{M}$ (here a will arise as the average of $\log w$); and locally bounded functions $c_1(y, r)$, $c_2(y, r)$, with $1 \leq c_1(y, r), c_2(y, r) < \infty$ for any $0 < r < \infty$, such that for all $B = B(y, r) \subset B_0$ the following two conditions hold

(1)

$$(5.47) \quad \operatorname{esssup}_{x \in \nu B} w_N \leq c_1(y, \nu r) e^{A(\ln \frac{1}{1-\nu})^\tau} \frac{1}{|B|} \int_B w_N$$

for every $0 < \nu_0 \leq \nu < 1$, $N \in \mathbb{N}$, where $w_N \equiv \Psi^{(-N)}(w)$, and

(2)

$$(5.48) \quad s |\{x \in B : \log w > s + a\}| < c_2(y, r) |B|$$

for every $s > 0$.

Then, for every ν with $0 < \nu_0 \leq \nu < 1$ there exists $b = b(\nu, \tau, c_1, c_2)$ such that

$$(5.49) \quad \operatorname{esssup}_{B(y, \nu r)} w \leq be^a.$$

More precisely, b is given by

$$b = \exp \left(C(\nu, A, \tau) c_2^*(r) c_1^*(r)^2 \right),$$

where

$$(5.50) \quad c_j^*(r) = c_j^*(y, r, \nu) = \sup_{\nu \leq s \leq 1} c_j(y, sr), \quad j = 1, 2.$$

and the constant $C(\nu, A, \tau)$ is bounded for $\tilde{\nu}$ away from 1.

PROOF. Define

$$\Omega(\rho) = \operatorname{esssup}_{x \in B(y, \rho)} (\log w(y) - a) \quad \text{for } \nu r \leq \rho \leq r.$$

First of all, we can rewrite the conclusion as $\Omega(\nu r) < C(\nu, A, \tau) c_2^*(r) c_1^*(r)^4$. If $\Omega(\nu r) \leq 2c_2^*(r)$ then estimate (5.49) holds with any $C(\nu, A, \tau) > 2$, therefore, we may assume $\Omega(\rho) > 2c_2^*(r) \geq 2c_2(\rho)$ for all $\nu r \leq \rho \leq r$. The idea is to find a recurrence inequality for $\Omega(r_k)$ with increasing radii $\nu r = r_0 < r_1 < r_2 < \dots < r_k < \dots < r$. Let us first fix a positive integer k and let $\nu_k = r_{k-1}/r_k \in (\nu_0, 1)$. We decompose the ball $B = B(y, r_k)$ in the following way

$$\begin{aligned} B &= B_1 \bigcup B_2 \\ &= \left\{ y \in B : \log w(y) - a > \frac{1}{2} \Omega(r_k) \right\} \cup \left\{ y \in B : \log w(y) - a \leq \frac{1}{2} \Omega(r_k) \right\}. \end{aligned}$$

For simplicity, we will write $c_j(y, r) = c_j(r)$, $j = 1, 2$. As in the proof of Lemma 49, we can write

$$\begin{aligned} \int_B \Psi^{(-N)}(u) &= \int_B \Psi^{(-N)}(e^{(\ln u - a) + a}) \\ &\leq \int_{B_1} \Psi^{(-N)}(e^{\Omega(r) + a}) + \int_{B_2} \Psi^{(-N)}(e^{\frac{\Omega(r)}{2} + a}) \\ &\leq \frac{2c_2(r)}{\Omega(r)} |B| \Psi^{(-N)}(e^{\Omega(r) + a}) + |B| \Psi^{(-N)}(e^{\frac{\Omega(r)}{2} + a}). \end{aligned}$$

Using (5.47) this gives

$$\Psi^{(-N)}(e^{\Omega(\nu r) + a}) \leq \phi(\nu) c_1(\nu r) \left(\frac{2c_2(r)}{\Omega(r)} \Psi^{(-N)}(e^{\Omega(r) + a}) + \Psi^{(-N)}(e^{\frac{\Omega(r)}{2} + a}) \right),$$

where we denote $\phi(\nu) = e^{A(\ln \frac{1}{1-\nu})^\tau}$.

We now choose N such that the two terms on the right in brackets are comparable, i.e.

$$(5.51) \quad \frac{1}{2} \Psi^{(-N)}(e^{\frac{\Omega(r)}{2} + a}) \leq \frac{2c_2(r)}{\Omega(r)} \Psi^{(-N)}(e^{\Omega(r) + a}) \leq \Psi^{(-N)}(e^{\frac{\Omega(r)}{2} + a}).$$

which can also be written as

$$(5.52) \quad \frac{\Psi^{(-N)}(e^{\Omega(r)+a})}{\Psi^{(-N)}(e^{\frac{\Omega(r)}{2}+a})} \approx \frac{\Omega(r)}{2c_2(r)}.$$

Recall that we assumed $2c_2(r)/\Omega(r) \leq 1$. Moreover, for any $x \in (0, 1/M)$ we have $\Psi^{(-N)}(x) \rightarrow 1/M$ as $N \rightarrow \infty$. This implies that the ratio on the left of (5.52) can be bounded above by 1 for all N large enough. On the other hand using the intermediate value theorem we can write

$$\frac{\Psi^{(-N)}(e^{\Omega(r)+a})}{\Psi^{(-N)}(e^{\frac{\Omega(r)}{2}+a})} = e^{\frac{d\Psi^{(-N)}(e^\xi)}{d\xi} \cdot \Omega(r)} = \frac{e^{\frac{d\Psi^{(-N)}(e^\xi)}{d\xi} \cdot \Omega(r)}}{\Omega(r)} \Omega(r)$$

with $\xi \in \left(\frac{\Omega(r)}{2} + a, \Omega(r) + a\right)$. We can calculate

$$\frac{d}{ds} \ln \Psi^{(-1)}(e^s) = \frac{d}{ds} \left((s - \ln A)^{\frac{1}{m}} + 1 \right)^m = \left(1 + \frac{1}{(s - \ln A)^{\frac{1}{m}}} \right)^{m-1} > \left(1 - \frac{1}{(\ln M + \ln A)^{\frac{1}{m}}} \right)^{m-1}$$

for $s < -\ln M$ which gives

$$\frac{e^{\frac{d\Psi^{(-1)}(e^\xi)}{d\xi} \cdot \Omega(r)}}{\Omega(r)} \geq \frac{e^{c\Omega(r)}}{\Omega(r)} \geq C.$$

We can therefore arrange to have

$$\frac{e^{\frac{d\Psi^{(-N)}(e^\xi)}{d\xi} \cdot \Omega(r)}}{\Omega(r)} \Omega(r) \gtrsim \frac{\Omega(r)}{c_2(r)}$$

by choosing an appropriate N and using $\Omega(r) \geq c_2(r)$. This concludes (5.52). There are now two cases to consider

Case 1:

$$C\phi(\nu)c_1(\nu r)\Psi^{(-N)}(e^{\frac{\Omega(r)}{2}+a}) \leq \Psi^{(-N)}(e^{\frac{3}{4}\Omega(r)+a})$$

This implies

$$\begin{aligned} \Psi^{(-N)}(e^{\Omega(\nu r)+a}) &\leq \Psi^{(-N)}(e^{\frac{3}{4}\Omega(r)+2a}) \\ \Omega(\nu r) &\leq \frac{3}{4}\Omega(r) \end{aligned}$$

Case 2:

$$C\phi(\nu)c_1(\nu r)\Psi^{(-N)}(e^{\frac{\Omega(r)}{2}+a}) > \Psi^{(-N)}(e^{\frac{3}{4}\Omega(r)+a})$$

which gives

$$(5.53) \quad C\phi(\nu)c_1(\nu r) \geq \frac{\Psi^{(-N)}(e^{\frac{3}{4}\Omega(r)+a})}{\Psi^{(-N)}(e^{\frac{\Omega(r)}{2}+a})}$$

We now note that $\ln \Psi^{(-1)}(e^s)$ is a concave function of s for $s < -\ln M$. We can calculate

$$\begin{aligned} \frac{d}{ds} \ln \Psi^{(-1)}(e^s) &= \frac{d}{ds} \left((s - \ln A)^{\frac{1}{m}} + 1 \right)^m = \left(1 + \frac{1}{(s - \ln A)^{\frac{1}{m}}} \right)^{m-1} \\ \frac{d^2}{ds^2} \ln \Psi^{(-1)}(e^s) &= -\frac{m-1}{m} \left(1 + \frac{1}{(s - \ln A)^{\frac{1}{m}}} \right)^{m-2} \frac{1}{(s - \ln A)^{\frac{m+1}{m}}} < 0 \end{aligned}$$

If we now write

$$\ln \Psi^{(-2)}(e^s) = \ln \Psi^{(-1)} \left(\Psi^{(-1)}(e^s) \right) = \ln \Psi^{(-1)} \left(e^{\ln \Psi^{(-1)}(e^s)} \right)$$

we see that $\ln \Psi^{(-2)}(e^s)$ is a composition of concave increasing functions. By induction we get that $\ln \Psi^{(-N)}(e^s)$ is a concave function of s for $s < -\ln M$. This in turn shows

$$\frac{\Psi^{(-N)}(e^{\frac{3}{4}\Omega(r)+a})}{\Psi^{(-N)}(e^{\frac{\Omega(r)}{2}+a})} \geq \frac{\Psi^{(-N)}(e^{\Omega(r)+a})}{\Psi^{(-N)}(e^{\frac{3}{4}\Omega(r)+a})}$$

and therefore

$$\left(\frac{\Psi^{(-N)}(e^{\frac{3}{4}\Omega(r)+a})}{\Psi^{(-N)}(e^{\frac{\Omega(r)}{2}+a})} \right)^2 \geq \frac{\Psi^{(-N)}(e^{\Omega(r)+a})}{\Psi^{(-N)}(e^{\frac{\Omega(r)}{2}+a})}.$$

Combining this with (5.53) and (5.51) we obtain

$$\Omega(r) \leq Cc_2(r) (\phi(\nu)c_1(\nu r))^2$$

Thus for each ν precisely one of the following happens

(1)

$$\Omega(\nu r) \leq \frac{3}{4}\Omega(r)$$

(2)

$$\Omega(\nu r) \leq Cc_2(r) (\phi(\nu)c_1(\nu r))^2$$

Recall that we chose a sequence of radii by the recurrence relation $r_0 = \nu r$, $r_k = r_{k-1}/\nu_k$, and we now assume that ν_k is generated by Lemma 60 below together with a constant $C = C(\nu, A, \tau, 4/3)$. Then from the above calculations, for each k we have at least one of the following holds:

(I) $\Omega(r_{k-1}) < \frac{3}{4}\Omega(r_k)$.

(II) $\Omega(r_{k-1}) < 16c_2(r) (\phi(\nu)c_1(r_{k-1}))^2$.

The condition (a) guarantees that $r_k < r$ for all k . Now let us consider the minimal positive integer n so that the inequality (II) above holds for $k = n$. First of all, if this integer does not exist, i.e. the condition (I) holds for all integers $k > 0$, then we have $\Omega(\nu r) = \Omega(r_0) < (3/4)^k \Omega(r_k)$ for all k , thus $\Omega(\nu r) \leq 0$ is a contradiction. On the other hand, if n is the minimal index mentioned above, then we can apply the inequality (I) for $k = 1, 2, \dots, n-1$ and the inequality (II) for $k = n$, and finally obtain

$$\Omega(\nu r) < (3/4)^{n-1} \Omega(r_{n-1}) < (3/4)^{n-1} \cdot 8c_2^*(r)c_1^*(r)^2 e^{4A(\ln \frac{1}{1-\nu_n})^\tau} = (32/3)C(\nu, A, \tau, 4/3)c_2^*(r)c_1^*(r)^2.$$

Here we have used condition (b) in the lemma below. ■

LEMMA 60. *Given constants $\nu \in (0, 1)$, $A, \tau > 0$ and $\eta > 1$, there exists a positive constant $C(\nu, A, \tau, \eta) > 1$ and a sequence $\{\nu_k\}_{k \in \mathbb{Z}^+}$, such that*

(a) $\nu_k \in (0, 1)$ and $\prod_{k=1}^{\infty} \nu_k > \nu$.

(b) $e^{4A(\ln \frac{1}{1-\nu_k})^\tau} = C(\nu, A, \tau, \eta) \cdot \eta^k$.

PROOF. The sequence is completely determined by the condition (II) above if the constant $C = C(\nu, A, \tau, \eta)$ has been chosen. Indeed a basic calculation shows

$$\nu_k = 1 - e^{-\left(\frac{k \ln \eta + \ln C}{4A}\right)^{1/\tau}} \in (0, 1).$$

Now we only need to find a constant C so that $\prod_{k=1}^{\infty} \nu_k > \nu$. Since we have the inequality

$$\prod_{k=1}^n (1 - x_k) > 1 - \sum_{k=1}^n x_k, \quad \text{whenever } x_k \in (0, 1),$$

it suffices to show

$$\sum_{k=1}^{\infty} e^{-\left(\frac{k \ln \eta + \ln C}{4A}\right)^{1/\tau}} < 1 - \nu.$$

This is always true for sufficiently large $C > 1$ by dominated convergence:

- The series $\sum_{k=1}^{\infty} e^{-\left(\frac{k \ln \eta + \ln C}{4A}\right)^{1/\tau}} < \sum_{k=1}^{\infty} e^{-\left(\frac{k \ln \eta}{4A}\right)^{1/\tau}} < \infty$.
- Each term satisfies $\lim_{C \rightarrow \infty} e^{-\left(\frac{k \ln \eta + \ln C}{4A}\right)^{1/\tau}} = 0$.

■

2.3. The supremum half of the Harnack inequality. We can now establish the other weak half Harnack inequality.

THEOREM 61. *Assume that $\varphi(r)$ and $\Phi(t) = \Psi_m(t)$ with $m > 2$ satisfy the Sobolev bump inequality (1.5), that the (1,1) Poincaré inequality (1.23) holds, and that a nonstandard sequence of Lipschitz cutoff functions exists. Let u be a nonnegative weak solution of $\mathcal{L}u = \phi$ in $B(y, r)$ with A -admissible ϕ . Then, for any ν such that $0 < \nu_0 \leq \nu < 1$, the weak solution u satisfies the following half Harnack inequality,*

$$(5.54) \quad \operatorname{esssup}_{x \in B(y, \nu r)} \left(u(x) + \|\phi\|_{X(B(y, r))} \right) \leq b e^{\langle \log(u + \|\phi\|_{X(B(y, r))}) \rangle_B},$$

with c_j^* as in (5.50),

$$C_{Har}(y, r, \nu) = b^2 = \exp \left(\frac{64c_1^*(r)^4 c_2^*(r)}{C(1-\nu)^{4\tau}} \right).$$

PROOF. By the Inner Ball inequality in Theorem 55 for $\Psi^{(-N)}\bar{u}$, with $\bar{u} = u + \|\phi\|_{X(B(y, r))}$ and $N \geq 1$, there exist a locally bounded function $c_1(y, r)$ and a constant τ such that for all $\nu_0 \leq \nu < 1$

$$(5.55) \quad \operatorname{esssup}_{x \in B(y, \nu r)} \Psi^{(-N)}\bar{u}(x) \leq c_1(y, \nu r) e^{A \left(\ln \frac{1}{1-\nu} \right)^\tau} \frac{1}{|B|} \int_B \Psi^{(-N)}\bar{u}.$$

Indeed, the constant $C(\varphi, m, r) = e^{C(\ln K)^m}$ can be written in the form

$$C(\varphi, m, r) = e^{C \left(\ln \frac{\varphi(r)}{\delta(r)} \right)^m} e^{C \left(\ln \frac{1}{1-\nu} \right)^m}$$

using the definition of K . Also, by Lemma 50 we have that there exists C_W such that for all $s > 0$

$$s |\{x \in B : \log \bar{u} > s + \langle \log \bar{u} \rangle_B\}| < C_W \frac{|B| r}{\delta_y(r)}.$$

Thus we may apply Lemma 59 to $w = \bar{u}$ with $c_2(y, r) = C_W r / \delta_y(r)$ and $a = \langle \log \bar{u} \rangle_B$ to obtain

$$(5.56) \quad \operatorname{esssup}_{B(y, \nu r)} (\bar{u}) \leq b e^{\langle \log \bar{u} \rangle_B}.$$

■

CONCLUSION 62. *If we have both (5.27), i.e.*

$$\frac{1}{b} e^{\langle \log \bar{u} \rangle_B} \leq \operatorname{ess\,inf}_{x \in B(y, \nu r)} \left(u(x) + \|\phi\|_{X(B(y, r))} \right),$$

and (5.54), i.e.

$$\operatorname{ess\,sup}_{x \in B(y, \nu r)} \left(u(x) + \|\phi\|_{X(B(y, r))} \right) \leq b e^{\langle \log \bar{u} \rangle_B},$$

then we have the weak Harnack inequality

$$\operatorname{ess\,sup}_{x \in B(y, \nu r)} \left(u(x) + \|\phi\|_{X(B(y, r))} \right) \leq C_{Har}(y, r, \nu) \operatorname{ess\,inf}_{x \in B(y, \nu r)} \left(u(x) + \|\phi\|_{X(B(y, r))} \right)$$

with

$$C_{Har}(y, r, \nu) = b^2 = \exp \left(\frac{64 c_1^*(r)^4 c_2^*(r)}{C(1-\nu)^{4\tau}} \right).$$

3. DeGiorgi iteration

We now recall the basic DeGiorgi argument from pages 84-86 of [SaWh4]. If we set $\omega(r)$ to be the oscillation of u on $B(0, r)$,

$$\omega(r) \equiv \operatorname{ess\,sup}_{x \in B(0, r)} u(x) - \operatorname{ess\,inf}_{x \in B(0, r)} u(x),$$

then $\omega(r)$ satisfies inequality (173) on page 85 of [SaWh4]:

$$(5.57) \quad \omega(c_2 r) \leq \left(1 - \frac{1}{2C_{Har}^{-1}} \right) \omega(r) + \sigma(r)$$

where c_2 is a small positive constant and where $\sigma(r)$ is a positive function vanishing at 0 which in the context of [SaWh4] is given by

$$\begin{aligned} \sigma(r) &= m(r) + N(r) \left(r^{2\eta} \|F\|_{\frac{q}{2}} + r^\eta \|\mathbf{G}\|_q \right) \\ &= \left(r^{2\eta} \|f\|_{\frac{q}{2}} + r^\eta \|\mathbf{g}\|_q \right) + N(r) \left(r^{2\eta} \|F\|_{\frac{q}{2}} + r^\eta \|\mathbf{G}\|_q \right). \end{aligned}$$

Here we will take

$$\sigma(r) = \|\phi\|_{X(B(0, r))},$$

with ϕ Dini admissible, so that $\sum_{k=0}^{\infty} \sigma(\tau^k r_0) = \sum_{k=0}^{\infty} \|\phi\|_{X(B(0, \tau^k r_0))} < \infty$.

REMARK 63. *This is the one place in this paper where we need to assume that ϕ is Dini admissible.*

The main lemma we use to deduce Hölder continuity from a Harnack inequality in the case C_{Har} is independent of $r > 0$, is a generalization of the DeGiorgi Lemma 8.23 in [GiTr] (see also Lemma 63 in [SaWh4]).

LEMMA 64. *Suppose $0 < \tau < 1$. Assume $\gamma : (0, R_0] \rightarrow (0, 1)$ and let ω, σ be non-negative, non-decreasing functions on $(0, R_0]$ so that*

$$\omega(\tau R) \leq \gamma(R) \omega(R) + \sigma(R), \quad 0 < R \leq R_0.$$

Suppose in addition that both

$$(5.58) \quad \prod_{k=0}^{\infty} \gamma(\tau^k R_0) = 0,$$

and

$$(5.59) \quad \sum_{k=0}^{\infty} \sigma(\tau^k R_0) < \infty.$$

Then

$$\lim_{R \rightarrow 0} \omega(R) = 0.$$

PROOF. The monotonicity and lower bound of the function ω guarantee the existence of the limit $\omega_0 = \lim_{R \rightarrow 0} \omega(R) = \inf_{0 < R \leq R_0} \omega(R)$. Suppose, in order to derive a contradiction, that $\omega_0 > 0$. By the recurrence formula, we have

$$\gamma(\tau^k R_0) \geq \frac{\omega(\tau^{k+1} R_0)}{\omega(\tau^k R_0)} \left[1 - \frac{\sigma(\tau^k R_0)}{\omega(\tau^{k+1} R_0)} \right] \geq \frac{\omega(\tau^{k+1} R_0)}{\omega(\tau^k R_0)} \left[1 - \frac{\sigma(\tau^k R_0)}{\omega_0} \right]$$

for all $k \geq 0$. Our assumption on σ implies in particular that $\sigma(\tau^k R_0) \rightarrow 0$, and therefore we have $\sigma(\tau^k R_0) < \omega_0$ for $k \geq k_0$ for a sufficiently large k_0 . Now by (5.58) we have

$$0 = \prod_{k=k_0}^{\infty} \gamma(\tau^k R_0) \geq \prod_{k=k_0}^{\infty} \frac{\omega(\tau^{k+1} R_0)}{\omega(\tau^k R_0)} \left[1 - \frac{\sigma(\tau^k R_0)}{\omega_0} \right] = \frac{\omega_0}{\omega(\tau^{k_0} R_0)} \prod_{k=k_0}^{\infty} \left[1 - \frac{\sigma(\tau^k R_0)}{\omega_0} \right],$$

which contradicts (5.59) since

$$\prod_{k=k_0}^{\infty} \left[1 - \frac{\sigma(\tau^k R_0)}{\omega_0} \right] = 0 \implies \sum_{k=k_0}^{\infty} \frac{\sigma(\tau^k R_0)}{\omega_0} = \infty.$$

by the well-known result that if $x_k \in [0, 1)$, then

$$\prod_{k=0}^{\infty} (1 - x_k) = 0 \iff \sum_{k=0}^{\infty} x_k = \infty.$$

■

From (5.57) we see that we may take the following choice of $\gamma(r)$ in the lemma above:

$$\gamma(r) = 1 - \frac{1}{2C_{Har}(r)}.$$

In our situation, the constant $C_{Har}(r)$ blows up as $r \rightarrow 0$, so that $\gamma(r) = 1 - \frac{1}{2C_{Har}(r)}$ tends to 1, and we must show that

$$(5.60) \quad \prod_{k=0}^{\infty} \gamma(\tau^k R_0) = 0.$$

This is equivalent to

$$\sum_{k=0}^{\infty} \frac{1}{2C_{Har}(\tau^k R_0)} = \sum_{k=0}^{\infty} (1 - \gamma(\tau^k R_0)) = \infty.$$

So for simplicity we take $R_0 = 1$ and

$$C_{Har}(r) = \exp \left(\left(\ln \ln \ln \frac{1}{r} \right) \left(\frac{1}{r} \right)^{4 \frac{\ln \ln \ln \frac{1}{r}}{\ln \ln \frac{1}{r}}} \right)$$

and compute that for $k > \ln \frac{1}{\tau}$ we have

$$\begin{aligned} C_{Har}(\tau^k) &= \exp \left(\left(\ln \ln \ln \frac{1}{\tau^k} \right) \left(\frac{1}{\tau^k} \right)^{4 \frac{\ln \ln \ln \frac{1}{\tau^k}}{\ln \ln \frac{1}{\tau^k}}} \right) \\ &= \exp \left(\left(\ln \left(\ln k + \ln \ln \frac{1}{\tau} \right) \right) e^{(\ln \frac{1}{\tau}) 4k \frac{\ln(\ln k + \ln \ln \frac{1}{\tau})}{(\ln k + \ln \ln \frac{1}{\tau})}} \right) \\ &> \exp \left(e^{4k \frac{\ln(\ln k)}{2 \ln k}} \right) \gg \exp(2k), \end{aligned}$$

so that $\sum \frac{1}{C_{Har}(\tau^k)} < \infty$. On the other hand, if we have $C_{Har}(r) \leq C \left(\ln \frac{1}{r} \right) \left(\ln \ln \frac{1}{r} \right)$, then

$$C_{Har}(\tau^k) = C \left(\ln \frac{1}{\tau^k} \right) \left(\ln \ln \frac{1}{\tau^k} \right) = C \left(k \ln \frac{1}{\tau} \right) \left(\ln k + \ln \ln \frac{1}{\tau} \right)$$

and

$$\sum_k \frac{1}{C_{Har}(\tau^k)} \approx \sum_k \frac{1}{C k \ln k} = \infty.$$

CONCLUSION 65. *If the Harnack constant $C_{Har}(r)$ satisfies*

$$(5.61) \quad C_{Har}(r) \leq C \left(\ln \frac{1}{r} \right) \left(\ln \ln \frac{1}{r} \right), \quad r \ll 1,$$

then $\sum_k \frac{1}{C_{Har}(\tau^k)} = \infty$ and continuity of weak solutions to $\mathcal{L}u = \phi$ holds provided $\|\phi\|_{X(B(0,r))} = o\left(\frac{1}{\ln \frac{1}{r}}\right)$. If however,

$$C_{Har}(r) \geq C \left(\ln \frac{1}{r} \right) \left(\ln \ln \frac{1}{r} \right)^{1+\varepsilon}, \quad r \ll 1,$$

then $\sum_k \frac{1}{C_{Har}(\tau^k)} < \infty$ and the method fails to yield continuity of weak solutions.

In order to complete the proof of Theorem 25, we need only show that (5.61) holds provided the doubling increment growth Condition 21 holds, and we now turn to proving this. Recall that the first three conditions imply the following Harnack inequality

$$\operatorname{esssup}_{x \in B(y, \nu r)} \left(u(x) + \|\phi\|_{X(B(y, r))} \right) \leq C_{Har}(y, r, \nu) \operatorname{essinf}_{x \in B(y, \nu r)} \left(u(x) + \|\phi\|_{X(B(y, r))} \right)$$

with

$$C_{Har}(y, r, \nu) = b^2 = \exp \left(\frac{64 c_1^*(r)^4 c_2^*(r)}{C(1-\nu)^{4\tau}} \right).$$

and

$$c_j^* = c_j^*(y, r, \tilde{\nu}) = \operatorname{esssup}_{\tilde{\nu} \leq s \leq 1} c_j(y, sr), \quad j = 1, 2.$$

The constants $c_1(r)$ and $c_2(r)$ satisfy the following estimates

$$\begin{aligned} c_1(r) &\approx e^{C(\ln K)^m} \approx e^{C'(\ln \frac{\varphi(r)}{\delta(r)})^m}, \\ c_2(r) &\approx C \frac{r}{\delta(r)}, \end{aligned}$$

which implies

$$(5.62) \quad C_{Har}(r) \approx e^{C c_2(r) c_1(r)^4} \approx \exp \left(C \frac{r}{\delta(r)} e^{C'(\ln \frac{\varphi(r)}{\delta(r)})^m} \right).$$

Recall that continuity of weak solutions is guaranteed by

$$C_{Har}(r) \leq C \left(\ln \frac{1}{r} \right) \left(\ln \ln \frac{1}{r} \right), \quad r \ll 1.$$

Combining this with (5.62) we obtain a condition on $\delta(r)$

$$C \frac{r}{\delta(r)} e^{C'(\ln \frac{\varphi(r)}{\delta(r)})^m} \leq \ln \ln \frac{1}{r}.$$

Without loss of generality we may assume that $\ln \frac{r}{\delta(r)} > 1$, then using that $m > 1$ and $\varphi(r) > r$, we conclude

$$C \left(\ln \frac{\varphi(r)}{\delta(r)} \right)^m \leq \ln^{(3)} \frac{1}{r}, \quad r \ll 1$$

where C is a large constant depending on m but independent of r . It is easy to see that the above condition is guaranteed by the growth (1.21) in the Introduction.

Part 3

Geometric results

In this third part of the paper, we turn to the problem of finding specific geometric conditions on the structure of our equations that permit us to prove the Orlicz Sobolev and Poincaré inequalities needed to apply the abstract theory in Part 2 above. The first chapter here deals with basic geometric estimates for a specific family of geometries, which are then applied in the next chapter to obtain the needed Orlicz Sobolev and Poincaré inequalities. Finally, in the third chapter in this part we prove our geometric theorems on local boundedness, the maximum principle and continuity of weak solutions.

Infinitely degenerate geometries in the plane

Here in this first chapter of the third part of the paper, we consider degenerate geometries in the plane, the properties of their geodesics and balls, and the associated subrepresentation inequalities. Recall from (1.8) that we are considering the inverse metric tensor given by the 2×2 diagonal matrix

$$A = \begin{bmatrix} 1 & 0 \\ 0 & f(x)^2 \end{bmatrix}.$$

Here the function $f(x)$ is an even twice continuously differentiable function on the real line \mathbb{R} with $f(0) = 0$ and $f'(x) > 0$ for all $x > 0$. The A -distance dt is given by

$$dt^2 = dx^2 + \frac{1}{f(x)^2} dy^2.$$

This distance coincides with the control distance as in [SaWh4], etc. since a vector v is subunit for an invertible symmetric matrix A , i.e. $(v \cdot \xi)^2 \leq \xi^{\text{tr}} A \xi$ for all ξ , if and only if $v^{\text{tr}} A^{-1} v \leq 1$. Indeed, if v is subunit for A , then

$$(v^{\text{tr}} A^{-1} v)^2 = (v \cdot A^{-1} v)^2 \leq (A^{-1} v)^{\text{tr}} A A^{-1} v = v^{\text{tr}} A^{-1} v,$$

and for the converse, Cauchy-Schwarz gives

$$(v \cdot \xi)^2 = (v^{\text{tr}} \xi)^2 = (v^{\text{tr}} A^{-1} A \xi)^2 \leq (v^{\text{tr}} A^{-1} A A^{-1} v) (\xi^{\text{tr}} A \xi) = (v^{\text{tr}} A^{-1} v) (\xi^{\text{tr}} A \xi).$$

1. Calculation of the A -geodesics

We now compute the equation satisfied by an A -geodesic γ passing through the origin. A geodesic minimizes the distance

$$\int_0^{x_0} \sqrt{1 + \frac{\left(\frac{dy}{dx}\right)^2}{f^2}} dx, \quad \text{where } (x, y) \text{ is on } \gamma,$$

and so the calculus of variations gives the equation

$$\frac{d}{dx} \left[\frac{\frac{dy}{dx}}{f^2 \sqrt{1 + \frac{\left(\frac{dy}{dx}\right)^2}{f^2}}} \right] = 0.$$

Consequently, the function

$$\lambda = \frac{f^2 \sqrt{1 + \frac{\left(\frac{dy}{dx}\right)^2}{f^2}}}{\frac{dy}{dx}}$$

is actually a positive constant conserved along the geodesic $y = y(x)$ that satisfies

$$\lambda^2 = \frac{f^2 \left[f^2 + \left(\frac{dy}{dx} \right)^2 \right]}{\left(\frac{dy}{dx} \right)^2} \implies (\lambda^2 - f^2) \left(\frac{dy}{dx} \right)^2 = f^4.$$

Thus if $\gamma_{0,\lambda}$ denotes the geodesic starting at the origin going in the vertical direction for $x > 0$, and parameterized by the constant λ , we have $\lambda = f(x)$ if and only if $\frac{dy}{dx} = \infty$. For convenience we temporarily assume that f is defined on $(0, \infty)$. Thus the geodesic $\gamma_{0,\lambda}$ turns back toward the y -axis at the unique point $(X(\lambda), Y(\lambda))$ on the geodesic where $\lambda = f(X(\lambda))$, provided of course that $\lambda < f(\infty) \equiv \sup_{x>0} f(x)$. On the other hand, if $\lambda > f(\infty)$, then $\frac{dy}{dx}$ is essentially constant for x large and the geodesics $\gamma_{0,\lambda}$ for $\lambda > f(\infty)$ look like straight lines with slope $\frac{f(\infty)^2}{\sqrt{\lambda^2 - f(\infty)^2}}$ for x large. Finally, if $\lambda = f(\infty)$, then the geodesic $\gamma_{0,\lambda}$ has slope that blows up at infinity.

DEFINITION 66. *We refer to the parameter λ as the turning parameter of the geodesic $\gamma_{0,\lambda}$, and to the point $T(\lambda) = (X(\lambda), Y(\lambda))$ as the turning point on the geodesic $\gamma_{0,\lambda}$.*

SUMMARY 67. *We summarize the turning behaviour of the geodesic $\gamma_{0,\lambda}$ as the turning parameter λ decreases from ∞ to 0:*

- (1) *When $\lambda = \infty$ the geodesic $\gamma_{0,\infty}$ is horizontal,*
- (2) *As λ decreases from ∞ to $f(\infty)$, the geodesics $\gamma_{0,\lambda}$ are asymptotically lines whose slopes increase to infinity,*
- (3) *At $\lambda = f(\infty)$ the geodesic $\gamma_{0,f(\infty)}$ has slope that increases to infinity as x increases,*
- (4) *As λ decreases from $f(\infty)$ to 0, the geodesics $\gamma_{0,\lambda}$ are turn back at $X(\lambda) = f^{-1}(\lambda)$, and return to the y -axis in a path symmetric about the line $y = Y(\lambda)$.*

Solving for $\frac{dy}{dx}$ we obtain the equation

$$\frac{dy}{dx} = \frac{\pm f^2(x)}{\sqrt{\lambda^2 - f(x)^2}}.$$

Thus the geodesic $\gamma_{0,\lambda}$ that starts from the origin going in the vertical direction for $x > 0$, and with turning parameter λ , is given by

$$y = \int_0^x \frac{f(u)^2}{\sqrt{\lambda^2 - f(u)^2}} du, \quad x > 0.$$

Since the metric is invariant under vertical translations, we see that the geodesic $\gamma_{\eta,\lambda}(t)$ whose lower point of intersection with the y -axis has coordinates $(0, \eta)$, and whose positive turning parameter is λ , is given by the equation

$$y = \eta + \int_0^x \frac{f(u)^2}{\sqrt{\lambda^2 - f(u)^2}} du, \quad x > 0.$$

Thus the entire family of A -geodesics in the right half plane is $\{\gamma_{\eta,\lambda}\}$ parameterized by $(\eta, \lambda) \in (-\infty, \infty) \times (0, \infty]$, where when $\lambda = \infty$, the geodesic $\gamma_{\eta,\infty}(t)$ is the horizontal line through the point $(0, \eta)$.

2. Calculation of A-arc length

Let dt denote A-arc length along the geodesic $\gamma_{0,\lambda}$ and let ds denote Euclidean arc length along $\gamma_{0,\lambda}$.

LEMMA 68. For $0 < x < X(\lambda)$ and (x, y) on the lower half of the geodesic $\gamma_{0,\lambda}$ we have

$$\begin{aligned}\frac{dy}{dx} &= \frac{f(x)^2}{\sqrt{\lambda^2 - f(x)^2}}, \\ \frac{dt}{dx}(x, y) &= \frac{\lambda}{\sqrt{\lambda^2 - f(x)^2}}, \\ \frac{dt}{ds}(x, y) &= \frac{\lambda}{\sqrt{\lambda^2 - f(x)^2} [1 - f(x)^2]}.\end{aligned}$$

PROOF. First we note that $y = \int_0^x \frac{f(u)^2}{\sqrt{\lambda^2 - f(u)^2}} du$ implies $\frac{dy}{dx} = \frac{f(x)^2}{\sqrt{\lambda^2 - f(x)^2}}$. Thus from $dt^2 = dx^2 + \frac{1}{f(x)^2} dy^2$ we have

$$\begin{aligned}\left(\frac{dt}{dx}\right)^2 &= 1 + \frac{1}{f(x)^2} \left(\frac{dy}{dx}\right)^2 = 1 + \frac{1}{f(x)^2} \left(\frac{f(x)^2}{\sqrt{\lambda^2 - f(x)^2}}\right)^2 \\ &= 1 + \frac{1}{f(x)^2} \frac{f(x)^4}{\lambda^2 - f(x)^2} = \frac{\lambda^2}{\lambda^2 - f(x)^2}.\end{aligned}$$

Then the density of t with respect to s at the point (x, y) on the lower half of the geodesic $\gamma_{0,\lambda}$ is given by

$$\begin{aligned}\frac{dt}{ds} &= \frac{\frac{dt}{dx}}{\frac{ds}{dx}} = \frac{\frac{\lambda}{\sqrt{\lambda^2 - f(x)^2}}}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}} = \frac{\frac{\lambda}{\sqrt{\lambda^2 - f(x)^2}}}{\sqrt{1 + \frac{f(x)^4}{\lambda^2 - f(x)^2}}} \\ &= \frac{\lambda}{\sqrt{(\lambda^2 - f(x)^2) \left(1 + \frac{f(x)^4}{\lambda^2 - f(x)^2}\right)}} = \frac{\lambda}{\sqrt{\lambda^2 - f(x)^2 + f(x)^4}} \\ &= \frac{\lambda}{\sqrt{\lambda^2 - f(x)^2} [1 - f(x)^2]}.\end{aligned}$$

■

Thus at the y -axis when $x = 0$, we have $\frac{dt}{ds} = 1$, and at the turning point $T(\lambda) = (X(\lambda), Y(\lambda))$ of the geodesic, when $\lambda^2 = f(x)^2$, we have $\frac{dt}{ds} = \frac{1}{\lambda} = \frac{1}{f(x)}$. This reflects the fact that near the y axis, the geodesic is nearly horizontal and so the metric arc length is close to Euclidean arc length; while at the turning point for λ small, the density of metric arc length is large compared to Euclidean arc length since movement in the vertical direction meets with much resistance when x is small.

In order to make precise estimates of arc length, we will need to assume some additional properties on the function $f(x)$ when $|x|$ is small.

Assumptions: Fix $R > 0$ and let $F(x) = -\ln f(x)$ for $0 < x < R$, so that

$$f(x) = e^{-F(|x|)}, \quad 0 < |x| < R.$$

We assume the following for some constants $C \geq 1$ and $\varepsilon > 0$:

- (1) $\lim_{x \rightarrow 0^+} F(x) = +\infty$;
- (2) $F'(x) < 0$ and $F''(x) > 0$ for all $x \in (0, R)$;
- (3) $\frac{1}{C} |F'(r)| \leq |F'(x)| \leq C |F'(r)|$, $\frac{1}{2}r < x < 2r < R$;
- (4) $\frac{1}{-xF'(x)}$ is increasing in the interval $(0, R)$ and satisfies $\frac{1}{-xF'(x)} \leq \frac{1}{\varepsilon}$ for $x \in (0, R)$;
- (5) $\frac{F''(x)}{-F'(x)} \approx \frac{1}{x}$ for $x \in (0, R)$.

These assumptions have the following consequences.

LEMMA 69. *Suppose that R , f and F are as above.*

- (1) *If $0 < x_1 < x_2 < R$, then we have*

$$F(x_1) > F(x_2) + \varepsilon \ln \frac{x_2}{x_1}, \text{ equivalently } f(x_1) < \left(\frac{x_1}{x_2}\right)^\varepsilon f(x_2).$$

- (2) *If $x_1, x_2 \in (0, R)$ and $\max \left\{ \varepsilon x_1, x_1 - \frac{1}{|F'(x_1)|} \right\} \leq x_2 \leq x_1 + \frac{1}{|F'(x_1)|}$, then we have*

$$\begin{aligned} |F'(x_1)| &\approx |F'(x_2)|, \\ f(x_1) &\approx f(x_2). \end{aligned}$$

- (3) *If $x \in (0, R)$, then we have*

$$\frac{F''(x)}{|F'(x)|^2} \approx \frac{1}{-xF'(x)} \lesssim 1.$$

PROOF. Assumptions (2) and (4) give $|F'(x_1)| > \frac{\varepsilon}{x}$, and so we have

$$F(x_1) - F(x_2) > \int_{x_1}^{x_2} \frac{\varepsilon}{x} dx = \varepsilon \ln \frac{x_2}{x_1},$$

which proves Part (1) of the lemma. Without loss of generality, assume now that $x_1 \leq x_2 \leq x_1 + \frac{1}{|F'(x_1)|}$. Then by Assumption (4) we also have $x_1 \leq x_2 \leq (1 + \frac{1}{\varepsilon})x_1$, and then by Assumption (3), the first assertion in Part (2) of the lemma holds, and with the bound,

$$\begin{aligned} F(x_1) - F(x_2) &= \int_{x_1}^{x_2} -F'(x) dx \\ &\approx |F'(x_1)| (x_2 - x_1) \leq 1. \end{aligned}$$

From this we get

$$1 \leq \frac{f(x_2)}{f(x_1)} = e^{F(x_1) - F(x_2)} \lesssim 1,$$

which proves the second assertion in Part (2) of the lemma. Finally, Assumptions (4) and (5) give

$$\frac{F''(x)}{|F'(x)|^2} = \frac{F''(x)}{-F'(x)} \frac{1}{-F'(x)} \approx \frac{1}{x} \frac{1}{-F'(x)} \lesssim 1,$$

which proves Part (3) of the lemma. ■

LEMMA 70. Suppose $\lambda > 0$, $0 < x < X(\lambda)$ and

$$y = \int_0^x \frac{f(u)^2}{\sqrt{\lambda^2 - f^2(u)}} du.$$

Then (x, y) lies on the lower half of the geodesic $\gamma_{0,\lambda}$ and

$$y \approx \frac{f(x)^2}{\lambda |F'(x)|}.$$

PROOF. Using first that $\frac{f(u)^2}{\sqrt{\lambda^2 - f^2(u)}}$ is increasing in u , and then that $F(u) = -\ln f(u)$, we have

$$y = \int_0^x \frac{f(u)^2}{\sqrt{\lambda^2 - f(u)^2}} du \approx \int_{\frac{x}{2}}^x \frac{f(u)^2}{\sqrt{\lambda^2 - f(u)^2}} du = \int_{\frac{x}{2}}^x \frac{1}{-2F'(u)} \frac{[f(u)^2]'}{\sqrt{\lambda^2 - f(u)^2}} du,$$

and then using Assumption (3) we get

$$y \approx \frac{1}{-F'(x)} \int_{\frac{x}{2}}^x \frac{[f(u)^2]'}{2\sqrt{\lambda^2 - f(u)^2}} du = \frac{1}{-F'(x)} \int_{f(\frac{x}{2})^2}^{f(x)^2} \frac{dv}{2\sqrt{\lambda^2 - v}}.$$

Now from Part (1) of Lemma 69 we obtain $f(\frac{x}{2})^2 < (\frac{1}{2})^{2\varepsilon} f(x)^2$ and so

$$y \approx \frac{1}{-F'(x)} \int_0^{f(x)^2} \frac{dv}{2\sqrt{\lambda^2 - v}} = \frac{\lambda - \sqrt{\lambda^2 - f(x)^2}}{-F'(x)} \approx \frac{f(x)^2}{\lambda |F'(x)|},$$

where the final estimate follows from $1 - \sqrt{1-t} = \frac{t}{1+\sqrt{1-t}} \approx t$, $0 < t < 1$, with $t = \frac{f(x)^2}{\lambda^2}$. ■

REMARK 71. We actually have the upper bound $y \leq \frac{f(x)^2}{\lambda |F'(x)|}$ since $F''(x) > 0$. Indeed, then $\frac{1}{-F'(x)}$ is increasing and for $f(x) < \lambda$ we have

$$\begin{aligned} y &= \int_0^x \frac{f(u)^2}{\sqrt{\lambda^2 - f(u)^2}} du = \int_0^x \frac{1}{-2F'(u)} \frac{[f(u)^2]'}{\sqrt{\lambda^2 - f(u)^2}} du \\ &\leq \frac{1}{-F'(x)} \int_0^x \frac{[f(u)^2]'}{2\sqrt{\lambda^2 - f(u)^2}} du = \frac{f(x)^2}{-\lambda F'(x)}. \end{aligned}$$

Now we can estimate the A-arc length of the geodesic $\gamma_{0,\lambda}$ between the two points $P_0 = (0, 0)$ and $P_1 = (x_1, y_1)$ where $0 < x_1 < X(\lambda)$ and

$$y_1 = \int_0^{x_1} \frac{f(u)^2}{\sqrt{\lambda^2 - f^2(u)}} du.$$

We have the formula

$$d(P_0, P_1) = \int_{P_0}^{P_1} dt = \int_{P_0}^{P_1} \frac{dt}{dx} dx = \int_0^{x_1} \frac{\lambda}{\sqrt{\lambda^2 - f(x)^2}} dx,$$

from which we obtain $x_1 < d(P_0, P_1)$.

LEMMA 72. *With notation as above we have*

$$\begin{aligned} x_1 &< d(P_0, P_1) \leq d((0, 0), (x_1, 0)) + d((x_1, 0), (x_1, y_1)) ; \\ d((0, 0), (x_1, 0)) &= x_1 , \\ d((x_1, 0), (x_1, y_1)) &\leq \frac{f(x_1)}{-\lambda F'(x_1)} \leq \frac{1}{-F'(x_1)} < \frac{1}{\varepsilon} x_1 . \end{aligned}$$

In particular we have $d(P_0, P_1) \approx x_1$.

PROOF. From Remark 71 we have

$$d((x_1, 0), (x_1, y_1)) \leq \frac{y_1}{f(x_1)} \leq \frac{f(x_1)}{-\lambda F'(x_1)},$$

and then we use $f(x_1) \leq \lambda$ and Assumption (4). ■

COROLLARY 73. $|F'(d(P_0, P_1))| \approx |F'(x_1)|$ and $f(d(P_0, P_1)) \approx f(x_1)$.

PROOF. Combine Part (2) of Lemma 69 with Lemma 72. ■

3. Integration over A-balls and Area

Here we investigate properties of the A-ball $B(0, r_0)$ centered at the origin 0 with radius $r_0 > 0$:

$$B(0, r_0) \equiv \{x \in \mathbb{R}^2 : d(0, x) < r_0\}, \quad r_0 > 0.$$

For this we will use ‘A-polar coordinates’ where $d(0, (x, y))$ plays the role of the radial variable, and the turning parameter λ plays the role of the angular coordinate. More precisely, given Cartesian coordinates (x, y) , the A-polar coordinates (r, λ) are given implicitly by the pair of equations

$$\begin{aligned} (6.1) \quad r &= \int_0^x \frac{\lambda}{\sqrt{\lambda^2 - f(u)^2}} du , \\ y &= \int_0^x \frac{f(u)^2}{\sqrt{\lambda^2 - f(u)^2}} du . \end{aligned}$$

In this section we will work out the change of variable formula for the quarter A-ball $QB(0, r_0)$.

DEFINITION 74. *Let $\lambda \in (0, \infty)$. The geodesic with turning parameter λ first moves to the right and then curls back at the turning point $T(\lambda) = (X(\lambda), Y(\lambda))$ when $x = X(\lambda) \equiv f^{-1}(\lambda)$. If $R(\lambda)$ denotes the A-arc length from the origin to the turning point $T(\lambda)$, we have*

$$\begin{aligned} R(\lambda) &\equiv d(0, T(\lambda)) = \int_0^{X(\lambda)} \frac{\lambda}{\sqrt{\lambda^2 - f(u)^2}} du, \\ Y(\lambda) &= \int_0^{X(\lambda)} \frac{f(u)^2}{\sqrt{\lambda^2 - f(u)^2}} du. \end{aligned}$$

The two parts of the geodesic $\gamma_{0,\lambda}$, cut at the point $T(\lambda)$, have different equations:

$$(6.2) \quad y = \begin{cases} \int_0^x \frac{f(u)^2}{\sqrt{\lambda^2 - f(u)^2}} du & \text{when } y \in [0, Y(\lambda)] \\ 2Y(\lambda) - \int_0^x \frac{f(u)^2}{\sqrt{\lambda^2 - f(u)^2}} du & \text{when } y \in [Y(\lambda), 2Y(\lambda)] \end{cases}.$$

We define the region covered by the first equation for the geodesics to be Region 1, and the region covered by the second equation for the geodesics to be Region 2. They are separated by the curve $y = Y(f(x))$. We now calculate the first derivative matrix $\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \lambda} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \lambda} \end{bmatrix}$ and the Jacobian $\frac{\partial(x,y)}{\partial(r,\lambda)}$ in Regions 1 and 2 separately.

3.1. Region 1. Applying implicit differentiation to the first equation in (6.1), we have

$$\begin{aligned} 1 &= \frac{\partial r}{\partial r} = \frac{\partial x}{\partial r} \cdot \frac{\lambda}{\sqrt{\lambda^2 - f(x)^2}}, \\ 0 &= \frac{\partial r}{\partial \lambda} = \frac{\partial x}{\partial \lambda} \cdot \frac{\lambda}{\sqrt{\lambda^2 - f(x)^2}} + \int_0^x \frac{\partial}{\partial \lambda} \left[\frac{\lambda}{\sqrt{\lambda^2 - f(u)^2}} \right] du, \end{aligned}$$

where

$$\frac{\partial}{\partial \lambda} \left[\frac{\lambda}{\sqrt{\lambda^2 - f(u)^2}} \right] = \frac{1 \cdot \sqrt{\lambda^2 - f(u)^2} - \lambda \cdot \frac{2\lambda}{2\sqrt{\lambda^2 - f(u)^2}}}{\lambda^2 - f(u)^2} = \frac{-f(u)^2}{(\lambda^2 - f(u)^2)^{\frac{3}{2}}}.$$

Thus we have

$$\begin{aligned} \frac{\partial x}{\partial r} &= \frac{\sqrt{\lambda^2 - f(x)^2}}{\lambda}, \\ \frac{\partial x}{\partial \lambda} &= \frac{\sqrt{\lambda^2 - f(x)^2}}{\lambda} \cdot \int_0^x \frac{f(u)^2}{(\lambda^2 - f(u)^2)^{\frac{3}{2}}} du. \end{aligned}$$

Applying implicit differentiation to the second equation in (6.1), we have

$$\begin{aligned} \frac{\partial y}{\partial r} &= \frac{\partial x}{\partial r} \cdot \frac{f(x)^2}{\sqrt{\lambda^2 - f(x)^2}}; \\ \frac{\partial y}{\partial \lambda} &= \frac{\partial x}{\partial \lambda} \cdot \frac{f(x)^2}{\sqrt{\lambda^2 - f(x)^2}} + \int_0^x \frac{\partial}{\partial \lambda} \left[\frac{f(u)^2}{\sqrt{\lambda^2 - f(u)^2}} \right] du \\ &= \frac{\partial x}{\partial \lambda} \cdot \frac{f(x)^2}{\sqrt{\lambda^2 - f(x)^2}} + \int_0^x \frac{-\lambda f(u)^2}{(\lambda^2 - f(u)^2)^{\frac{3}{2}}} du \end{aligned}$$

Plugging the equation for $\frac{\partial x}{\partial \lambda}$ into these equations, we obtain

$$\begin{aligned}\frac{\partial y}{\partial r} &= \frac{f(x)^2}{\lambda}, \\ \frac{\partial y}{\partial \lambda} &= \frac{f(x)^2 - \lambda^2}{\lambda} \cdot \int_0^x \frac{f(u)^2}{(\lambda^2 - f(u)^2)^{\frac{3}{2}}} du,\end{aligned}$$

and this completes the calculation of the first derivative matrix $\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \lambda} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \lambda} \end{bmatrix}$.

Now we can calculate the Jacobian

$$\begin{aligned}\frac{\partial(x, y)}{\partial(r, \lambda)} &= \det \begin{bmatrix} \frac{\sqrt{\lambda^2 - f(x)^2}}{\lambda} & \frac{\sqrt{\lambda^2 - f(x)^2}}{\lambda} \cdot \int_0^x \frac{f(u)^2}{(\lambda^2 - f(u)^2)^{\frac{3}{2}}} du \\ \frac{f(x)^2}{\lambda} & \frac{f(x)^2 - \lambda^2}{\lambda} \cdot \int_0^x \frac{f(u)^2}{(\lambda^2 - f(u)^2)^{\frac{3}{2}}} du \end{bmatrix} \\ &= -\sqrt{\lambda^2 - f(x)^2} \int_0^x \frac{f(u)^2}{(\lambda^2 - f(u)^2)^{\frac{3}{2}}} du.\end{aligned}$$

In addition we have

$$\int_0^x \frac{f(u)^2}{(\lambda^2 - f(u)^2)^{\frac{3}{2}}} du \approx \int_{x/2}^x \frac{f(u)^2}{(\lambda^2 - f(u)^2)^{\frac{3}{2}}} du = \int_{x/2}^x \frac{\frac{d}{du} [f(u)^2] \cdot \frac{f(u)}{2f'(u)}}{(\lambda^2 - f(u)^2)^{\frac{3}{2}}} du,$$

where

$$\frac{f(u)}{2f'(u)} = \frac{1}{-2F'(u)} \approx \frac{1}{-F'(x)},$$

and so we have

$$\int_0^x \frac{f(u)^2}{(\lambda^2 - f(u)^2)^{\frac{3}{2}}} du \approx \frac{1}{-F'(x)} \int_{f(x/2)^2}^{f(x)^2} \frac{1}{(\lambda^2 - v)^{\frac{3}{2}}} dv.$$

By Part (1) of Lemma 69, we have $f(x/2) < (\frac{1}{2})^\varepsilon f(x)$, and as a result, we obtain

$$\int_0^x \frac{f(u)^2}{(\lambda^2 - f(u)^2)^{\frac{3}{2}}} du \approx \frac{1}{-F'(x)} \int_0^{f(x)^2} \frac{1}{(\lambda^2 - v)^{\frac{3}{2}}} dv \approx \frac{1}{-F'(x)} \left(\frac{1}{\sqrt{\lambda^2 - f(x)^2}} - \frac{1}{\lambda} \right).$$

Altogether we have the estimate

$$(6.3) \quad \left| \frac{\partial(x, y)}{\partial(r, \lambda)} \right| \approx \frac{1}{-F'(x)} \cdot \frac{\lambda - \sqrt{\lambda^2 - f(x)^2}}{\lambda} \approx \frac{f(x)^2}{\lambda^2 |F'(x)|}.$$

From Corollary 73, we also have

$$(6.4) \quad \left| \frac{\partial(x, y)}{\partial(r, \lambda)} \right| \approx \frac{f(r)^2}{\lambda^2 |F'(r)|}.$$

3.2. Region 2. In Region 2 we have the following pair of formulas:

$$(6.5) \quad \begin{aligned} r &= 2R(\lambda) - \int_0^x \frac{\lambda}{\sqrt{\lambda^2 - f(u)^2}} du, \\ y &= 2Y(\lambda) - \int_0^x \frac{f(u)^2}{\sqrt{\lambda^2 - f(u)^2}} du. \end{aligned}$$

where we recall that $R(\lambda) = \int_0^{X(\lambda)} \frac{\lambda}{\sqrt{\lambda^2 - f(u)^2}} du$ is the arc length of the geodesic $\gamma_{0,\lambda}$ from the origin 0 to the turning point $T(\lambda)$. Before proceeding, we calculate the derivative of $Y(\lambda)$. We note that due to cancellation, the derivative $R'(\lambda)$ does not explicitly enter into the formula for the Jacobian $\frac{\partial(x,y)}{\partial(r,\lambda)}$ below, so we defer its calculation for now.

LEMMA 75. *The derivative of $Y(\lambda)$ is given by*

$$Y'(\lambda) = \int_0^{f^{-1}(\lambda)} \frac{F''(u)}{|F'(u)|^2} \frac{\lambda}{\sqrt{\lambda^2 - f(u)^2}} du.$$

PROOF. Integrating by parts we obtain

$$\begin{aligned} Y(\lambda) &= \int_0^{f^{-1}(\lambda)} \frac{-f(u)}{f'(u)} \cdot \frac{d}{du} \sqrt{\lambda^2 - f(u)^2} du \\ &= \int_0^{f^{-1}(\lambda)} \frac{1}{F'(u)} \cdot \frac{d}{du} \sqrt{\lambda^2 - f(u)^2} du \\ &= - \int_0^{f^{-1}(\lambda)} \sqrt{\lambda^2 - f(u)^2} \cdot \frac{d}{du} \frac{1}{F'(u)} du \\ &= \int_0^{f^{-1}(\lambda)} \frac{F''(u)}{|F'(u)|^2} \sqrt{\lambda^2 - f(u)^2} du, \end{aligned}$$

and so from $\lambda^2 - f(f^{-1}(\lambda))^2 = 0$, we have

$$Y'(\lambda) = 0 + \int_0^{f^{-1}(\lambda)} \frac{F''(u)}{|F'(u)|^2} \frac{\lambda}{\sqrt{\lambda^2 - f(u)^2}} du.$$

■

Applying implicit differentiation to the first equation in (6.5), we have

$$\begin{aligned} 1 &= \frac{\partial r}{\partial r} = - \frac{\partial x}{\partial r} \cdot \frac{\lambda}{\sqrt{\lambda^2 - f(x)^2}}, \\ 0 &= \frac{\partial r}{\partial \lambda} = 2R'(\lambda) - \frac{\partial x}{\partial \lambda} \cdot \frac{\lambda}{\sqrt{\lambda^2 - f(x)^2}} - \int_0^x \frac{\partial}{\partial \lambda} \left[\frac{\lambda}{\sqrt{\lambda^2 - f(u)^2}} \right] du, \end{aligned}$$

where

$$\frac{\partial}{\partial \lambda} \left[\frac{\lambda}{\sqrt{\lambda^2 - f(u)^2}} \right] = \frac{1 \cdot \sqrt{\lambda^2 - f(u)^2} - \lambda \cdot \frac{2\lambda}{2\sqrt{\lambda^2 - f(u)^2}}}{\lambda^2 - f(u)^2} = \frac{-f(u)^2}{\left(\lambda^2 - f(u)^2\right)^{\frac{3}{2}}}.$$

Thus we have

$$\begin{aligned} \frac{\partial x}{\partial r} &= -\frac{\sqrt{\lambda^2 - f(x)^2}}{\lambda}, \\ \frac{\partial x}{\partial \lambda} &= \frac{2\sqrt{\lambda^2 - f(x)^2}}{\lambda} L'(\lambda) + \frac{\sqrt{\lambda^2 - f(x)^2}}{\lambda} \cdot \int_0^x \frac{f(u)^2}{\left(\lambda^2 - f(u)^2\right)^{\frac{3}{2}}} du. \end{aligned}$$

Applying implicit differentiation to the second equation in (6.5), we have

$$\begin{aligned} \frac{\partial y}{\partial r} &= -\frac{\partial x}{\partial r} \cdot \frac{f(x)^2}{\sqrt{\lambda^2 - f(x)^2}}; \\ \frac{\partial y}{\partial \lambda} &= 2Y'(\lambda) - \frac{\partial x}{\partial \lambda} \cdot \frac{f(x)^2}{\sqrt{\lambda^2 - f(x)^2}} - \int_0^x \frac{\partial}{\partial \lambda} \left[\frac{f(u)^2}{\sqrt{\lambda^2 - f(u)^2}} \right] du \\ &= 2Y'(\lambda) - \frac{\partial x}{\partial \lambda} \cdot \frac{f(x)^2}{\sqrt{\lambda^2 - f(x)^2}} - \int_0^x \frac{-\lambda f(u)^2}{\left(\lambda^2 - f(u)^2\right)^{\frac{3}{2}}} du \end{aligned}$$

Plugging the equation for $\frac{\partial x}{\partial \lambda}$ above into these equations, we have

$$\begin{aligned} \frac{\partial y}{\partial r} &= \frac{f(x)^2}{\lambda}, \\ \frac{\partial y}{\partial \lambda} &= 2Y'(\lambda) - \frac{2f(x)^2}{\lambda} R'(\lambda) + \frac{\lambda^2 - f(x)^2}{\lambda} \cdot \int_0^x \frac{f(u)^2}{\left(\lambda^2 - f(u)^2\right)^{\frac{3}{2}}} du. \end{aligned}$$

Thus the Jacobian is given by

$$\begin{aligned}
\frac{\partial(x, y)}{\partial(r, \lambda)} &= \det \begin{bmatrix} -\frac{\sqrt{\lambda^2 - f(x)^2}}{\lambda} & \frac{2\sqrt{\lambda^2 - f(x)^2}}{\lambda} R'(\lambda) + \frac{\sqrt{\lambda^2 - f(x)^2}}{\lambda} \cdot \int_0^x \frac{f(u)^2}{(\lambda^2 - f(u)^2)^{\frac{3}{2}}} du \\ \frac{f(x)^2}{\lambda} & 2Y'(\lambda) - \frac{2f(x)^2}{\lambda} R'(\lambda) + \frac{\lambda^2 - f(x)^2}{\lambda} \cdot \int_0^x \frac{f(u)^2}{(\lambda^2 - f(u)^2)^{\frac{3}{2}}} du \end{bmatrix} \\
&= \det \begin{bmatrix} -\frac{\sqrt{\lambda^2 - f(x)^2}}{\lambda} & \frac{\sqrt{\lambda^2 - f(x)^2}}{\lambda} \cdot \int_0^x \frac{f(u)^2}{(\lambda^2 - f(u)^2)^{\frac{3}{2}}} du \\ \frac{f(x)^2}{\lambda} & 2Y'(\lambda) + \frac{\lambda^2 - f(x)^2}{\lambda} \cdot \int_0^x \frac{f(u)^2}{(\lambda^2 - f(u)^2)^{\frac{3}{2}}} du \end{bmatrix} \\
&= -\sqrt{\lambda^2 - f(x)^2} \left\{ \int_0^x \frac{f(u)^2}{(\lambda^2 - f(u)^2)^{\frac{3}{2}}} du + \frac{2}{\lambda} Y'(\lambda) \right\} \\
&= -\sqrt{\lambda^2 - f(x)^2} \left\{ \int_0^x \frac{f(u)^2}{(\lambda^2 - f(u)^2)^{\frac{3}{2}}} du + \frac{2}{\lambda} \int_0^{f^{-1}(\lambda)} \frac{F''(u)}{|F'(u)|^2} \frac{\lambda}{\sqrt{\lambda^2 - f(u)^2}} du \right\} \\
&= -\sqrt{\lambda^2 - f(x)^2} \left\{ \int_0^x \frac{f(u)^2}{(\lambda^2 - f(u)^2)^{\frac{3}{2}}} du + \int_0^{f^{-1}(\lambda)} \frac{F''(u)}{|F'(u)|^2} \cdot \frac{2}{\sqrt{\lambda^2 - f(u)^2}} du \right\}.
\end{aligned}$$

In fact, we have

$$\begin{aligned}
\int_0^x \frac{f(u)^2}{(\lambda^2 - f(u)^2)^{\frac{3}{2}}} du &= \int_0^x \frac{f(u)}{f'(u)} \cdot \frac{d}{du} \left[\frac{1}{\sqrt{\lambda^2 - f(u)^2}} \right] du \\
&= \int_0^x \frac{1}{-F'(u)} \cdot \frac{d}{du} \left[\frac{1}{\sqrt{\lambda^2 - f(u)^2}} \right] du \\
&= \frac{1}{-F'(x)} \cdot \frac{1}{\sqrt{\lambda^2 - f(x)^2}} - \int_0^x \frac{1}{\sqrt{\lambda^2 - f(u)^2}} \cdot \frac{d}{du} \left[\frac{1}{-F'(u)} \right] du \\
&= \frac{1}{-F'(x)} \cdot \frac{1}{\sqrt{\lambda^2 - f(x)^2}} - \int_0^x \frac{1}{\sqrt{\lambda^2 - f(u)^2}} \cdot \frac{F''(u)}{|F'(u)|^2} du
\end{aligned}$$

As a result, we have within a factor of 2,

$$\begin{aligned}
\left| \frac{\partial(x, y)}{\partial(r, \lambda)} \right| &\approx \sqrt{\lambda^2 - f(x)^2} \left\{ \frac{1}{-F'(x)} \cdot \frac{1}{\sqrt{\lambda^2 - f(x)^2}} + \int_0^{f^{-1}(\lambda)} \frac{F''(u)}{|F'(u)|^2} \cdot \frac{1}{\sqrt{\lambda^2 - f(u)^2}} du \right\} \\
(6.6) \quad &= \frac{1}{-F'(x)} + \sqrt{\lambda^2 - f(x)^2} \int_0^{f^{-1}(\lambda)} \frac{F''(u)}{|F'(u)|^2} \cdot \frac{1}{\sqrt{\lambda^2 - f(u)^2}} du.
\end{aligned}$$

By Assumption (5), we have

$$\int_0^{f^{-1}(\lambda)} \frac{F''(u)}{|F'(u)|^2} \cdot \frac{1}{\sqrt{\lambda^2 - f(u)^2}} du \approx \int_0^{f^{-1}(\lambda)} \frac{1}{-uF'(u)} \cdot \frac{1}{\sqrt{\lambda^2 - f(u)^2}} du.$$

By Assumptions (3) and (4), the function $\frac{1}{-uF'(u)}$ increases and satisfies the doubling property, and so

$$\begin{aligned} \int_0^{f^{-1}(\lambda)} \frac{F''(u)}{|F'(u)|^2} \cdot \frac{1}{\sqrt{\lambda^2 - f(u)^2}} du &\approx \frac{1}{-f^{-1}(\lambda)F'(f^{-1}(\lambda))} \int_0^{f^{-1}(\lambda)} \frac{1}{\sqrt{\lambda^2 - f(u)^2}} du \\ &= \frac{1}{-f^{-1}(\lambda)F'(f^{-1}(\lambda))} \frac{R(\lambda)}{\lambda} \\ (6.7) \quad &\simeq \frac{1}{-\lambda F'(f^{-1}(\lambda))} \end{aligned}$$

since $R(\lambda) \approx f^{-1}(\lambda)$ by Lemma 72. Finally we can combine (6.6) and (6.7) to obtain

$$\left| \frac{\partial(x, y)}{\partial(r, \lambda)} \right| \approx \frac{1}{-F'(x)} + \frac{\sqrt{\lambda^2 - f(x)^2}}{\lambda} \cdot \frac{1}{-F'(f^{-1}(\lambda))} \approx \frac{1}{-F'(f^{-1}(\lambda))}.$$

According to Corollary 73, we also have

$$\left| \frac{\partial(x, y)}{\partial(r, \lambda)} \right| \approx \frac{1}{-F'(R(\lambda))}.$$

3.3. Integral of Radial Functions. Summarizing our estimates on the Jacobian we have

$$\left| \frac{\partial(x, y)}{\partial(r, \lambda)} \right| \approx \begin{cases} \frac{f(r)^2}{\lambda^2 |F'(r)|} \simeq \frac{f(x)^2}{\lambda^2 |F'(x)|} & \text{when } r < R(\lambda) \\ \frac{1}{|F'(f^{-1}(\lambda))|} \simeq \frac{1}{|F'(R(\lambda))|} & \text{when } R(\lambda) < r < 2R(\lambda) \end{cases}.$$

Therefore we have the following change of variable formula for nonnegative functions w :

$$\begin{aligned} \iint_{QB(0, r_0)} w dx dy &= \int_0^{r_0} \left[\int_{R^{-1}(\frac{r}{2})}^{\infty} w \left| \frac{\partial(x, y)}{\partial(r, \lambda)} \right| d\lambda \right] dr \\ &\approx \int_0^{r_0} \left[\int_{R^{-1}(\frac{r}{2})}^{R^{-1}(r)} w(r, \lambda) \frac{1}{|F'(R(\lambda))|} d\lambda + \int_{R^{-1}(r)}^{\infty} w(r, \lambda) \frac{f(r)^2}{\lambda^2 |F'(r)|} d\lambda \right] dr \\ &\approx \int_0^{r_0} \left[\int_{R^{-1}(\frac{r}{2})}^{R^{-1}(r)} w(r, \lambda) \frac{1}{|F'(r)|} d\lambda + \int_{R^{-1}(r)}^{\infty} w(r, \lambda) \frac{f^2(r)}{\lambda^2 |F'(r)|} d\lambda \right] dr \end{aligned}$$

If w is a radial function, then we have

$$\begin{aligned} \iint_{QB(0, r_0)} w dx dy &\approx \int_0^{r_0} w(r) \left[\int_{R^{-1}(\frac{r}{2})}^{R^{-1}(r)} \frac{1}{|F'(r)|} d\lambda + \int_{R^{-1}(r)}^{\infty} \frac{f(r)^2}{\lambda^2 |F'(r)|} d\lambda \right] dr \\ &\approx \int_0^{r_0} w(r) \left[\frac{R^{-1}(r) - R^{-1}(\frac{r}{2})}{|F'(r)|} + \frac{f(r)^2}{R^{-1}(r) |F'(r)|} \right] dr. \end{aligned}$$

From Corollary 73, we have $R^{-1}(r) \simeq f(r)$, and so we have

$$(6.8) \quad \iint_{B(0, r_0)} w(r) dx dy \approx \int_0^{r_0} w(r) \frac{f(r)}{|F'(r)|} dr.$$

CONCLUSION 76. *The area of the A-ball $B(0, r_0)$ satisfies*

$$(6.9) \quad \text{Area}(B(0, r_0)) = \iint_{B(0, r_0)} dx dy \approx \int_0^{r_0} \frac{f(r)}{|F'(r)|} dr \approx \frac{f(r_0)}{|F'(r_0)|^2}.$$

PROOF. Since $F(r) = -\ln f(r)$, we have $F'(r) = -\frac{f'(r)}{f(r)}$ and $\frac{f(r)}{-F'(r)} = \frac{f(r)^2}{f'(r)} = \frac{f(r)^2}{f'(r)^2} f'(r) = \frac{f'(r)}{|F'(r)|^2}$, and so

$$\begin{aligned} \iint_{B(0, r_0)} dx dy &\approx \int_0^{r_0} \frac{f(r)}{|F'(r)|} dr \approx \int_{\frac{r_0}{2}}^{r_0} \frac{f(r)}{|F'(r)|} dr = \int_{\frac{r_0}{2}}^{r_0} \frac{f'(r)}{|F'(r)|^2} dr \\ &\approx \frac{1}{|F'(r_0)|^2} \int_{\frac{r_0}{2}}^{r_0} f'(r) dr = \frac{f(r_0) - f(\frac{r_0}{2})}{|F'(r_0)|^2} \approx \frac{f(r_0)}{|F'(r_0)|^2}. \end{aligned}$$

■

3.4. Balls centered at an arbitrary point. In this section we consider the “height” of an arbitrary A-ball and its relative position in the ball. Let $X = (x_1, 0)$ be a point on the positive x -axis and let r be a positive real number. Let the upper half of the boundary of the ball $B(X, r)$ be given as the graph of the function $\varphi(x)$, $x_1 - r < x < x_1 + r$. Denote by $\beta_{X,P}$ the geodesic that meets the boundary of the ball $B(X, r)$ at the point $P = (x_1 + r^*, h)$ where $\beta_{X,P}$ has a vertical tangent at P , $r^* = r^*(x_1, r)$ and $h = h(x_1, r) = \varphi(x_1 + r^*)$. Here both r^* and h are *functions* of the two independent variables x_1 and r , but we will often write $r^* = r^*(x_1, r)$ and $h = h(x_1, r)$ for convenience.

PROPOSITION 77. *Let $\beta_{X,P}$, r^* and h be defined as above. Define $\lambda(x)$ implicitly by*

$$r = \int_{x_1}^x \frac{\lambda(u)}{\sqrt{\lambda(u)^2 - f(u)^2}} du.$$

Then

- (1) *For $x_1 - r < x < x_1 + r$ we have $\varphi(x) \leq \varphi(x_1 + r^*) = h$.*
- (2) *If $r \geq \frac{1}{|F'(x_1)|}$, then*

$$h \approx \frac{f(x_1 + r)}{|F'(x_1 + r)|} \text{ and } r - r^* \approx \frac{1}{|F'(x_1 + r)|}.$$

- (3) *If $r \leq \frac{1}{|F'(x_1)|}$, then*

$$h \approx r f(x_1) \text{ and } r - r^* \approx r.$$

We begin by proving part (1) of Proposition 77. First consider the case $x \geq x_1 + r^*$. Then we are in Region 1 and so $\lambda(x) \geq f(x)$ and we have

$$\varphi(x) = \int_{x_1}^x \frac{f(u)^2}{\sqrt{\lambda(u)^2 - f(u)^2}} du.$$

Differentiating $\varphi(x)$ we get

$$\varphi'(x) = \frac{f(x)^2}{\sqrt{\lambda(x)^2 - f(x)^2}} - \left(\int_{x_1}^x \frac{f(u)^2}{(\lambda(x)^2 - f(u)^2)^{\frac{3}{2}}} du \right) \lambda(x) \lambda'(x),$$

and differentiating the definition of $\lambda(x)$ implicitly gives

$$0 = \frac{\lambda(x)}{\sqrt{\lambda(x)^2 - f(x)^2}} - \left(\int_{x_1}^x \frac{f(u)^2}{(\lambda(x)^2 - f(u)^2)^{\frac{3}{2}}} du \right) \lambda'(x).$$

Combining equalities yields

$$\varphi'(x) = \frac{f(x)^2}{\sqrt{\lambda(x)^2 - f(x)^2}} - \lambda(x) \frac{\lambda(x)}{\sqrt{\lambda(x)^2 - f(x)^2}} = -\sqrt{\lambda(x)^2 - f(x)^2}.$$

When $x = x_1 + r^*$ we have $\infty = \frac{dy}{dx} = \frac{f(x)^2}{\sqrt{\lambda(x)^2 - f(x)^2}}$, which implies $\lambda(x) = f(x)$, and hence $\varphi'(x_1 + r^*) = 0$. Thus we have $\varphi(x) \leq \varphi(x_1 + r^*) = h$ for $x \geq x_1 + r^*$. Similar arguments show that $\varphi(x) \leq \varphi(x_1 + r^*) = h$ for $x_1 - r \leq x < x_1 + r^*$, and this completes the proof of part (1).

Now we turn to the proofs of parts (2) and (3) of Proposition 77. The locus (x, y) of the geodesic $\beta_{X,P}$ satisfies

$$(6.10) \quad y = \int_{x_1}^x \frac{f(u)^2}{\sqrt{(\lambda^*)^2 - f(u)^2}} du,$$

where $\lambda^* = f(x_1 + r^*)$. We will use the following two lemmas in the proofs of parts (2) and (3) of Proposition 77.

LEMMA 78. *The height $h = h(x_1, r)$ and the horizontal displacement $r - r^* = r - r^*(x_1, r)$ satisfy*

$$f(x_1 + r^*) \cdot (r - r^*) \leq h \leq 2f(x_1 + r^*) \cdot (r - r^*).$$

PROOF. The A -arc length r of $\beta_{X,P}$ is given by

$$r = \int_{x_1}^{x_1 + r^*} \frac{\lambda^*}{\sqrt{(\lambda^*)^2 - f(u)^2}} du.$$

Thus

$$r - r^* = \int_{x_1}^{x_1 + r^*} \left(\frac{\lambda^*}{\sqrt{(\lambda^*)^2 - f(u)^2}} - 1 \right) du = \int_{x_1}^{x_1 + r^*} \frac{f(u)^2}{\sqrt{(\lambda^*)^2 - f(u)^2}} \cdot \frac{1}{\lambda^* + \sqrt{(\lambda^*)^2 - f(u)^2}} du$$

Comparing this with the height $h = \int_{x_1}^{x_1 + r^*} \frac{f^2(u)}{\sqrt{(\lambda^*)^2 - f(u)^2}} du$, we have

$$\frac{h}{2\lambda^*} \leq r - r^* \leq \frac{h}{\lambda^*}.$$

This completes the proof since $\lambda^* = f(x_1 + r^*)$. ■

LEMMA 79. *The height h satisfies the estimate*

$$h \approx \frac{1}{|F'(x_1 + r^*)|} \sqrt{f(x_1 + r^*)^2 - f(x_1)^2}.$$

In fact the right hand is an exact upper bound:

$$h \leq \frac{1}{|F'(x_1 + r^*)|} \sqrt{f(x_1 + r^*)^2 - f(x_1)^2}.$$

PROOF. Using the fact that $\frac{1}{-F'(u)} = \frac{1}{|F'(u)|}$ is increasing, together with the equation (6.10) for the geodesic $\beta_{X,P}$, we have

$$\begin{aligned} h(x_1, r) &= \int_{x_1}^{x_1 + r^*(x_1, r)} \frac{f(u)^2}{\sqrt{(\lambda^*)^2 - f(u)^2}} du = \int_{x_1}^{x_1 + r^*} \frac{\frac{d}{du} [f(u)^2]}{\sqrt{(\lambda^*)^2 - f(u)^2}} \cdot \frac{1}{-2F'(u)} du \\ &\leq \frac{1}{|F'(x_1 + r^*)|} \int_{x_1}^{x_1 + r^*} \frac{\frac{d}{du} [f(u)^2]}{2\sqrt{(\lambda^*)^2 - f(u)^2}} du \\ &= \frac{1}{|F'(x_1 + r^*)|} \sqrt{f(x_1 + r^*)^2 - f(x_1)^2}, \end{aligned}$$

where in the last line we used $\lambda^* = f(x_1 + r^*)$. To prove the reverse estimate, we consider two cases:

Case 1: If $r^* < x_1$, then we use our assumption that $\frac{1}{-F'(u)} = \frac{1}{|F'(u)|}$ has the doubling property to obtain

$$\begin{aligned} h &= \int_{x_1}^{x_1 + r^*} \frac{f(u)^2}{\sqrt{(\lambda^*)^2 - f(u)^2}} du = \int_{x_1}^{x_1 + r^*} \frac{\frac{d}{du} [f(u)^2]}{\sqrt{(\lambda^*)^2 - f(u)^2}} \cdot \frac{1}{-2F'(u)} du \\ &\simeq \frac{1}{|F'(x_1 + r^*)|} \int_{x_1}^{x_1 + r^*} \frac{\frac{d}{du} [f(u)^2]}{2\sqrt{(\lambda^*)^2 - f(u)^2}} du \\ &= \frac{1}{|F'(x_1 + r^*)|} \sqrt{f(x_1 + r^*)^2 - f(x_1)^2}. \end{aligned}$$

Case 2: If $r^* \geq x_1$, we make a similar estimate by modifying the lower limit of integral, and using the fact that $f(u)$ increases:

$$\begin{aligned} h &\approx \int_{x_1 + \frac{r^*}{2}}^{x_1 + r^*} \frac{f(u)^2}{\sqrt{(\lambda^*)^2 - f(u)^2}} du = \int_{x_1 + \frac{r^*}{2}}^{x_1 + r^*} \frac{\frac{d}{du} [f(u)^2]}{\sqrt{(\lambda^*)^2 - f(u)^2}} \cdot \frac{1}{-2F'(u)} du \\ &\approx \frac{1}{|F'(x_1 + r^*)|} \int_{x_1 + \frac{r^*}{2}}^{x_1 + r^*} \frac{\frac{d}{du} [f(u)^2]}{2\sqrt{(\lambda^*)^2 - f(u)^2}} du \\ &= \frac{1}{|F'(x_1 + r^*)|} \sqrt{f(x_1 + r^*)^2 - f\left(x_1 + \frac{r^*}{2}\right)^2}. \end{aligned}$$

Finally we have

$$\sqrt{f(x_1 + r^*)^2 - f\left(x_1 + \frac{r^*}{2}\right)^2} \approx f(x_1 + r^*) \approx \sqrt{f(x_1 + r^*)^2 - f(x_1)^2}$$

by the assumption $r^* \geq x_1$ together with Part 1 of Lemma 69.

This completes the proof of Lemma 79. ■

COROLLARY 80. *Combining Lemmas 78 and 79, for $h = h(x_1, r)$ and $r^* = r^*(x_1, r)$, we have*

$$f(x_1 + r^*) \cdot (r - r^*) \leq h \leq \frac{1}{|F'(x_1 + r^*)|} \sqrt{f(x_1 + r^*)^2 - f(x_1)^2},$$

and as a result,

$$(6.11) \quad r - r^* \leq \frac{1}{|F'(x_1 + r^*)|} \cdot \frac{\sqrt{f(x_1 + r^*)^2 - f(x_1)^2}}{f(x_1 + r^*)} \leq \frac{1}{|F'(x_1 + r^*)|}.$$

From part (2) of Lemma 69 we obtain

$$(6.12) \quad \begin{aligned} |F'(x_1 + r)| &\approx |F'(x_1 + r^*)|, \\ f(x_1 + r) &\simeq f(x_1 + r^*). \end{aligned}$$

We now split the proof of Proposition 77 into two cases.

3.4.1. *Proof of part (2) of Proposition 77 for $r \geq \frac{1}{|F'(x_1)|}$.* By Lemmas 78 and 79, we have

$$(6.13) \quad r - r^*(x_1, r) = r - r^* \approx \frac{1}{|F'(x_1 + r^*)|} \cdot \frac{\sqrt{f(x_1 + r^*)^2 - f(x_1)^2}}{f(x_1 + r^*)}.$$

We consider two cases.

Case A: If $r^* > r_1 \equiv \frac{1}{2|F'(x_1)|}$, then we have

$$F(x_1) - F(x_1 + r^*) = \int_{x_1}^{x_1 + r^*} |F'(x_1)| dx \geq \int_{x_1}^{x_1 + r_1} |F'(x_1)| dx \geq |F'(x_1 + r_1)| \cdot r_1 \gtrsim 1.$$

Here we used the estimate $|F'(x_1 + r_1)| \approx |F'(x_1)|$ given by Part 2 of Lemma 69. This implies

$$\ln \frac{f(x_1 + r^*)}{f(x_1)} \gtrsim 1,$$

and we have

$$\frac{\sqrt{f(x_1 + r^*)^2 - f(x_1)^2}}{f(x_1 + r^*)} \approx 1.$$

Plugging this into (6.13), we have $r - r^* \approx \frac{1}{|F'(x_1 + r^*)|}$. The proof is completed using (6.12) and Lemma 78.

Case B: If $r^* \leq r_1$, then we have $|F'(x_1 + r^*)| \approx |F'(x_1)|$ and $r - r^* \geq \frac{1}{2|F'(x_1)|}$. Therefore we have

$$r - r^* \gtrsim \frac{1}{|F'(x_1 + r^*)|}.$$

Combining this with (6.11), we obtain $r - r^* \approx \frac{1}{|F'(x_1 + r^*)|}$ again, and the proof is completed as in the first case.

3.4.2. *Proof of part (3) of Proposition 77 for $r \leq \frac{1}{|F'(x_1)|}$.* In this case Lemma 78 and Lemma 79 give with $r^* = r^*(x_1, r)$,

$$\begin{aligned} f(x_1 + r^*) \cdot (r - r^*) &\approx h(x_1, r) \approx \frac{1}{|F'(x_1 + r^*)|} \sqrt{f(x_1 + r^*)^2 - f(x_1)^2} \\ &\approx \frac{1}{|F'(x_1)|} \left(\int_{x_1}^{x_1 + r^*} 2f(u)^2 |F'(u)| du \right)^{\frac{1}{2}} \\ &\approx \frac{1}{|F'(x_1)|} \left[2f(x_1 + r^*)^2 |F'(x_1)| \cdot r^* \right]^{\frac{1}{2}} \\ &\approx \frac{\sqrt{r^*} f(x_1 + r^*)}{\sqrt{|F'(x_1)|}}, \end{aligned}$$

where we have used Part 2 of Lemma 69 and the fact $r^* = r^*(x_1, r) < r \leq \frac{1}{|F'(x_1)|}$. This implies

$$[|F'(x_1)| (r - r^*)]^2 \approx |F'(x_1)| r^*.$$

Thus

$$[|F'(x_1)| (r - r^*)]^2 + |F'(x_1)| (r - r^*) \approx |F'(x_1)| r \leq 1.$$

As a result, we have $|F'(x_1)| (r - r^*) \approx |F'(x_1)| r \implies r - r^* \approx r$. This also gives the estimate for h by Lemma 78 since we already have $f(x_1 + r^*) \approx f(x_1)$.

3.5. Area of balls centered at an arbitrary point. In the following proposition we obtain an estimate, similar to (6.9), for areas of balls centered at arbitrary points.

PROPOSITION 81. *Let $P = (x_1, x_2) \in \mathbb{R}^2$ and $r > 0$. Set*

$$B_+(P, r) \equiv \{(y_1, y_2) \in B(P, r) : y_1 > x_1 + r^*\}.$$

If $r \geq \frac{1}{|F'(x_1)|}$ then we recover (6.9)

$$|B(P, r)| \approx \frac{f(x_1 + r)}{|F'(x_1 + r)|^2} \approx |B_+(P, r)|.$$

On the other hand, if $r \leq \frac{1}{|F'(x_1)|}$ we have

$$|B(P, r)| \approx r^2 f(x_1) \approx |B_+(P, r)|$$

PROOF. Because of symmetry, it is enough to consider $x_1 > 0$ and $y_1 = 0$. So let $P_1 = (x_1, 0)$ with $x_1 > 0$.

Case $r \geq \frac{1}{|F'(x_1)|}$. In this case we will compare the ball $B(P, r)$ to the ball $B(0, R)$ centered at the origin with radius $R = x_1 + r$. First we note that $B(P, r) \subset B(0, R)$ since if $(x, y) \in B(P, r)$, then

$$\text{dist}((0, 0), (x, y)) \leq \text{dist}((0, 0), P) + \text{dist}(P, (x, y)) < x_1 + r = R.$$

Thus from (6.9) we have

$$|B(P, r)| \leq |B(0, R)| \approx \frac{f(R)}{|F'(R)|^2} = \frac{f(x_1 + r)}{|F'(x_1 + r)|^2},$$

By parts (2) and (3) of Proposition 77, $h \approx \frac{f(x_1 + r)}{|F'(x_1 + r)|}$ and $r - r^* \approx \frac{1}{|F'(x_1 + r)|}$ when $r \geq \frac{1}{|F'(x_1)|}$, and so we have

$$|B(P, r)| \lesssim h(x_1, r) (r - r^*(x_1, r)).$$

Finally, we claim that

$$h(x_1, r) (r - r^*(x_1, r)) \lesssim |B(P, r)|.$$

To see this we consider x satisfying $x_1 + r^* \leq x \leq x_1 + \frac{r+r^*}{2}$, where $x_1 + \frac{r+r^*}{2}$ is the midpoint of the interval $[x_1 + r^*, x_1 + r]$ corresponding to the "thick" part of the ball $B(P, r)$. For such x we let $y > 0$ be defined so that $(x, y) \in \partial B(P, r)$. Then using the taxicab path $(x_1, 0) \rightarrow (x, 0) \rightarrow (x, y)$, we see that

$$(6.14) \quad x - x_1 + \frac{y}{f(x)} \geq \text{dist}((x_1, 0), (x, y)) = r,$$

implies

$$y \geq f(x) (r - x + x_1) \approx f(x_1 + r) (r - r^*),$$

where the final approximation follows from $r - r^* \approx \frac{1}{|F'(x_1+r)|}$ and part (2) of Lemma 69 upon using $x_1 + r^* \leq x \leq x_1 + \frac{r+r^*}{2}$. Thus, using parts (2) and (3) of Proposition 77 again, we obtain

$$\begin{aligned} |B(P, r)| &\gtrsim [f(x_1 + r) (r - r^*)] (r - r^*) \\ &\approx \frac{f(x_1 + r)}{|F'(x_1 + r)|} (r - r^*) \approx h(x_1, r) (r - r^*(x_1, r)), \end{aligned}$$

which proves our claim and concludes the proof that $|B(P, r)| \approx \frac{f(x_1+r)}{|F'(x_1+r)|^2} \approx |B_+(P, r)|$ when $r \geq \frac{1}{|F'(x_1)|}$.

Case $r < \frac{1}{|F'(x_1)|}$. In this case parts (2) and (3) of Proposition 77 show that $h \approx rf(x_1)$ and $r - r^* \approx r$, and part (1) shows that h maximizes the 'height' of the ball. Thus we immediately obtain the upper bound

$$|B(P, r)| \lesssim hr \lesssim f(x_1) r^2.$$

To obtain the corresponding lower bound, we use notation as in the first case and note that (6.14) now implies

$$(6.15) \quad y \geq f(x) (r - x + x_1) \approx f(x_1) r,$$

where the final approximation follows from part (2) of Proposition 77 and part (2) of Lemma 69 upon using $x_1 + r^* \leq x \leq x_1 + \frac{r+r^*}{2}$. Thus

$$|B(P, r)| \gtrsim [f(x_1) r] (r - r^*) \approx f(x_1) r^2,$$

which concludes the proof that $|B(P, r)| \approx f(x_1) r^2 \approx |B_+(P, r)|$ when $r < \frac{1}{|F'(x_1)|}$. ■

Using Proposition 77 we obtain a useful corollary for the measure of the "thick" part of a ball. But first we need to establish that $r^*(x_1, r)$ is increasing in r where

$$T(x_1, r) \equiv (x_1 + r^*(x_1, r), h(x_1, r))$$

is the turning point for the geodesic γ_r that passes through $P = (x_1, 0)$ in the upward direction and has vertical slope at the boundary of the ball $B(P, r)$.

LEMMA 82. *Let $x_1 > 0$. Then $r^*(x_1, r') < r^*(x_1, r)$ if $0 < r' < r$.*

PROOF. Let $T(x_1, r) \equiv (x_1 + r^*(x_1, r), h(x_1, r))$ be the turning point for the geodesic γ_r that passes through $P = (x_1, 0)$ and has vertical slope at the boundary of the ball $B(P, r)$. A key property of this geodesic is that it continues beyond the point $T(x_1, r)$ by vertical reflection. Now we claim that this key property implies that when $0 < r' < r$, the geodesic $\gamma_{r'}$ cannot lie below γ_r just to the right of P . Indeed, if it did, then since $B(P, r') \subset B(P, r)$ implies

$h(x_1, r') < h(x_1, r)$, the geodesic $\gamma_{r'}$ would turn back and intersect γ_r in the first quadrant, contradicting the fact that geodesics cannot intersect twice in the first quadrant. Thus the geodesic $\gamma_{r'}$ lies above γ_r just to the right of P , and it is now evident that $\gamma_{r'}$ must turn back ‘before’ γ_r , i.e. that $r^*(x_1, r') < r^*(x_1, r)$. ■

COROLLARY 83. *Denote*

$$\begin{aligned} B_+(P, r) &\equiv \{(y_1, y_2) \in B(P, r) : y_1 > x_1 + r^*\}, \\ B_-(P, r) &\equiv \{(y_1, y_2) \in B(P, r) : y_1 \leq x_1 + r^*\}. \end{aligned}$$

Then

$$|B_+(P, r)| \approx |B_-(P, r)| \approx |B(P, r)|.$$

PROOF. **Case** $r < \frac{1}{|F'(x_1)|}$. Recall from Assumption (4) that $\frac{1}{|F'(x_1)|} \leq \frac{1}{\varepsilon}x_1$, so that in this case we have $r < \frac{1}{\varepsilon}x_1$, and hence also that $x_1 - \max\{\varepsilon x_1, x_1 - \frac{r}{2}\} \approx r$. From Proposition 81 we have

$$|B(P, r)| \approx r^2 f(x_1).$$

From part (2) of Lemma 69, there is a positive constant c such that $f(x) \geq cf(x_1)$ for $\max\{\varepsilon x_1, x_1 - \frac{r}{2}\} \leq x \leq x_1$. It follows that $B_-(P, r) \supset (\max\{\varepsilon x_1, x_1 - \frac{r}{2}\}, x_1) \times (-\frac{c}{2}f(x_1)r, \frac{c}{2}f(x_1)r)$ since

$$\begin{aligned} d((x_1, 0), (x, y)) &\leq d((x_1, 0), (x, 0)) + d((x, 0), (x, y)) \\ &= |x_1 - x| + \frac{|y|}{f(x)} < \frac{r}{2} + \frac{r}{2} = r, \end{aligned}$$

provided $\max\{\varepsilon x_1, x_1 - \frac{r}{2}\} < x < x_1$ and $-\frac{c}{2}f(x_1)r < y < \frac{c}{2}f(x_1)r$. Thus we have

$$|B_-(P, r)| \geq cr^2 f(x_1).$$

Case $r \geq \frac{1}{|F'(x_1)|}$. The bound $|B_-(P, r)| \leq |B(P, r)| \approx |B_+(P, r)|$ follows from Proposition 81. We now consider two subcases in order to obtain the lower bound $|B_-(P, r)| \gtrsim |B(P, r)|$.

Subcase $r \geq \frac{1}{|F'(x_1)|} \geq r^*$. By (6.11) and part (2) of Lemma 69 we have

$$|F'(x_1 + r)| \approx |F'(x_1 + r^*)| \text{ and } f(x_1 + r) \approx f(x_1 + r^*).$$

Then by Proposition 81, followed by the above inequalities, and then another application of part (2) of Lemma 69, we obtain

$$|B(P, r)| \approx \frac{f(x_1 + r)}{|F'(x_1 + r)|^2} \approx \frac{f(x_1 + r^*)}{|F'(x_1 + r^*)|^2} \approx \frac{f(x_1)}{|F'(x_1)|^2}.$$

On the other hand, with $r_0 = \frac{1}{|F'(x_1)|}$, we can apply the case already proved above, together with the fact that $m(x_1, r)$ is increasing in r , to obtain that

$$\begin{aligned} |B_-(P, r)| &\geq |\{(y_1, y_2) \in B(P, r_0) : y_1 \leq x_1 + r^*\}| \\ &\geq |\{(y_1, y_2) \in B(P, r_0) : y_1 \leq x_1 + (r_0)^*\}| \approx \frac{f(x_1)}{|F'(x_1)|^2}. \end{aligned}$$

Subcase $r \geq r^* \geq \frac{1}{|F'(x_1)|}$. Since $B(P, r^*) \subset B_-(P, r)$ we can apply Proposition 81 to $B(P, r^*)$ to obtain

$$|B_-(P, r)| \geq |B(P, r^*)| \approx \frac{f(x_1 + r^*)}{|F'(x_1 + r^*)|^2}.$$

Now we apply (6.12) and Proposition 81 again to conclude that

$$\frac{f(x_1 + r^*)}{|F'(x_1 + r^*)|^2} \approx \frac{f(x_1 + r)}{|F'(x_1 + r)|^2} \approx |B(P, r)|.$$

■

CHAPTER 7

Orlicz norm Sobolev and Poincaré inequalities in the plane

Here in this chapter, we prove Sobolev and Poincaré inequalities for infinitely degenerate geometries in the plane. The key to these inequalities is a subrepresentation formula whose kernel in the infinitely degenerate setting is in general much smaller than the familiar $\frac{\text{distance}}{\text{volume}}$ kernel that arises in the finite type case.

1. Subrepresentation inequalities

We will obtain a subrepresentation formula for the degenerate geometry by applying the method of Lemma 79 in [SaWh4]. For simplicity, we will only consider x with $x_1 > 0$; since our metric is symmetric about the y axis it suffices to consider this case. For the general case, all objects defined on the right half plane must be defined on the left half plane by reflection about the y -axis.

Consider a sequence of metric balls $\{B(x, r_k)\}_{k=1}^\infty$ centered at x with radii $r_k \searrow 0$ such that $r_0 = r$ and

$$|B(x, r_k) \setminus B(x, r_{k+1})| \approx |B(x, r_{k+1})|, \quad k \geq 1,$$

so that $B(x, r_k)$ is divided into two parts having comparable area. We may in fact assume that

$$(7.1) \quad r_{k+1} = \begin{cases} r^*(x_1, r_k) & \text{if } r_k \geq \frac{1}{|F'(x_1)|} \\ \frac{1}{2}r_k & \text{if } r_k < \frac{1}{|F'(x_1)|} \end{cases}$$

where r^* is defined in Proposition 77. Indeed, if $r_k \geq \frac{1}{|F'(x_1)|}$, then by (1) in Proposition 77 we have that

$$r_k - r_{k+1} \approx \frac{1}{|F'(x_1 + r_k)|}$$

and then by (2) in Lemma 69 it follows that $f(x_1 + r_k) \approx f(x_1 + r_{k+1})$ and $|F'(x_1 + r_k)| \approx |F'(x_1 + r_{k+1})|$, so by Corollary 83 and (1) in Proposition 77 it follows that

$$\begin{aligned} |B(x, r_k)| &\approx \left| \left\{ B(x, r_k) \cap \{y_1 > x_1 + r_{k+1}\} \right\} \right| \approx (r_k - r_{k+1}) h(x_1, x_1 + r_k) \\ &\approx \frac{1}{|F'(x_1 + r_k)|} \frac{f(x_1 + r_k)}{|F'(x_1 + r_k)|} \approx \frac{1}{|F'(x_1 + r_{k+1})|} \frac{f(x_1 + r_{k+1})}{|F'(x_1 + r_{k+1})|} \\ &\approx (r_{k+1} - r_{k+2}) h(x_1, x_1 + r_{k+1}) \approx \left| \left\{ B(x, r_{k+1}) \cap \{y_1 > x_1 + r_{k+2}\} \right\} \right| \\ &\approx |B(x, r_{k+1})|. \end{aligned}$$

On the other hand, if $r_k \leq \frac{1}{|F'(x_1)|}$ then by (2) in Proposition 77 $r_k - r_{k+1} \approx r_k$ and $h(x_1, x_1 + r_k) \approx r_k f(x_1)$, hence by Corollary 83

$$|B(x, r_k)| \approx (r_k - r_{k+1}) h(x_1, x_1 + r_{k+1}) \approx r_k^2 f(x_1) \approx r_{k+1}^2 f(x_1) \approx |B(x, r_{k+1})|.$$

As a consequence we also have that

$$(r_k - r_{k+1}) h(x_1, x_1 + r_k) \approx (r_{k+1} - r_{k+2}) h(x_1, x_1 + r_{k+1}) \lesssim (r_{k+1} - r_{k+2}) h(x_1, x_1 + r_k)$$

so $r_k - r_{k+1} \leq C(r_{k+1} - r_{k+2}) \leq Cr_{k+1}$, which yields

$$(7.2) \quad \frac{1}{C+1} r_k \leq r_{k+1}.$$

Now for $x_1, t > 0$ define

$$h^*(x_1, t) = \int_{x_1}^{x_1+t} \frac{f^2(u)}{\sqrt{f^2(x_1+t) - f^2(u)}} du,$$

so that $h^*(x_1, t)$ describes the ‘height’ above x_2 at which the geodesic through $x = (x_1, x_2)$ curls back toward the y -axis at the point $(x_1 + t, x_2 + h^*(x_1, t))$. Thus the graph of $y = h^*(x_1, t)$ is the curve separating the analogues of Region 1 and Region 2 relative to the ball $B(x, r)$. Then in the case $r_k \geq \frac{1}{|F'(x_1)|}$, we have $h^*(x_1, r_{k+1}) = h(x_1, r_k)$, $k \geq 0$, where $h(x_1, r_k)$ is the height of $B(x, r_k)$ as defined in Proposition 77. In the opposite case $r_k < \frac{1}{|F'(x_1)|}$, we have $r_{k+1} = \frac{1}{2}r_k$ instead, and we will estimate differently.

For $k \geq 0$ define

$$E(x, r_k) \equiv \begin{cases} \{y : x_1 + r_{k+1} \leq y_1 < x_1 + r_k, |y_2| < h^*(x_1, y_1 - x_1)\} & \text{if } r_k \geq \frac{1}{|F'(x_1)|} \\ \{y : x_1 + r_{k+1} \leq y_1 < x_1 + r_k, |y_2| < h^*(x_1, r_k^*) = h(x_1, r_k)\} & \text{if } r_k < \frac{1}{|F'(x_1)|} \end{cases},$$

where we have written $r_k^* = r^*(x_1, r_k)$ for convenience. We claim that

$$(7.3) \quad |E(x, r_k)| \approx \left| E(x, r_k) \cap B(x, r_k) \right| \approx |B(x, r_k)| \text{ for all } k \geq 1.$$

Indeed, in the first case $r_k \geq \frac{1}{|F'(x_1)|}$, the second set of inequalities follows immediately by Corollary 83, and since $E(x, r_k) \subset B(x, r_{k-1})$ we have that

$$\begin{aligned} \left| E(x, r_k) \cap B(x, r_k) \right| &\leq |E(x, r_k)| \leq |B(x, r_{k-1})| \\ &\lesssim |B(x, r_k)| \lesssim \left| E(x, r_k) \cap B(x, r_k) \right|, \end{aligned}$$

which establishes the first set of inequalities in (7.3). In the second case $r_k < \frac{1}{|F'(x_1)|}$, we have

$$|E(x, r_k)| = \frac{1}{2} r_k h^*(x_1, r_k^*) \approx (r_k - r_k^*) h(x_1, r_k^*) \approx |B(x, r_k)|,$$

and from (6.15) with $(x, y) \in \partial B(x, r_k)$, we have

$$y \geq f(x)(r_k - x + x_1) \approx f(x_1) r_k,$$

for all $x \in [x_1, x_1 + r]$ since we are in the case $r_k < \frac{1}{|F'(x_1)|}$. It follows that

$$E(x, r_k) \cap B(x, r_k) \supset \left[x_1 + \frac{r_k}{2}, x_1 + \frac{3r_k}{4} \right] \times [-cf(x_1)r_k, cf(x_1)r_k]$$

and hence that

$$|E(x, r_k) \cap B(x, r_k)| \geq \frac{1}{2} cr_k f(x_1) r_k \approx |B(x, r_k)| \geq |E(x, r_k) \cap B(x, r_k)|.$$

This completes the proof of (7.3).

Now define $\Gamma(x, r)$ to be the set

$$\Gamma(x, r) = \bigcup_{k=1}^{\infty} E(x, r_k).$$

LEMMA 84. *With notation as above, in particular with $r_0 = r$ and r_1 given by (7.1), and assuming $\int_{E(x, r_1)} w = 0$, we have the subrepresentation formula*

$$(7.4) \quad w(x) \leq C \int_{\Gamma(x, r)} |\nabla_A w(y)| \frac{\widehat{d}(x, y)}{|B(x, d(x, y))|} dy,$$

where ∇_A is as in (1.10) and

$$\widehat{d}(x, y) \equiv \min \left\{ d(x, y), \frac{1}{|F'(x_1 + d(x, y))|} \right\}.$$

Note that when $f(r) = r^N$ is finite type, then $\widehat{d}(x, y) \approx d(x, y)$.

PROOF. Recall the sequence $\{r_k\}_{k=1}^{\infty}$ of decreasing radii above. Then since w is *a priori* Lipschitz continuous, and $\int_{E(x, r_1)} w = 0$, we can write

$$\begin{aligned} w(x) &= \lim_{k \rightarrow \infty} \frac{1}{|E(x, r_k)|} \int_{E(x, r_k)} w(y) dy \\ &= \sum_{k=1}^{\infty} \left\{ \frac{1}{|E(x, r_{k+1})|} \int_{E(x, r_{k+1})} w(y) dy - \frac{1}{|E(x, r_k)|} \int_{E(x, r_k)} w(z) dz \right\}, \end{aligned}$$

and so we have

$$\begin{aligned} |w(x)| &\lesssim \sum_{k=1}^{\infty} \frac{1}{|B(x, r_k)|^2} \int_{E(x, r_{k+1}) \times E(x, r_k)} |w(y) - w(z)| dy dz \\ &\lesssim \sum_{k=1}^{\infty} \frac{1}{|B(x, r_k)|^2} \int_{E(x, r_{k+1}) \times E(x, r_k)} \times \{|w(y_1, y_2) - w(z_1, y_2)| + |w(z_1, y_2) - w(z_1, z_2)|\} dy dz \\ &\lesssim \sum_{k=1}^{\infty} \frac{1}{|B(x, r_k)|^2} \int_{E(x, r_{k+1}) \times E(x, r_k)} \int_{y_1}^{z_1} |w_x(s, y_2)| ds dy_1 dy_2 dz_1 dz_2 \\ &\quad + \sum_{k=1}^{\infty} \frac{1}{|B(x, r_k)|^2} \int_{E(x, r_{k+1}) \times E(x, r_k)} \int_{y_2}^{z_2} |w_y(z_1, t)| dt dy_1 dy_2 dz_1 dz_2, \end{aligned}$$

which, with $H_k(x) \equiv E(x, r_{k+1}) \cup E(x, r_k)$, is dominated by

$$\begin{aligned} &\sum_{k=1}^{\infty} \frac{1}{|B(x, r_k)|^2} \left(\int_{H_k(x)} |\nabla_A w(s, y_2)| ds dy_2 \right) \{r_k - r_{k+1}\} \int_{H_k(x)} dz_1 dz_2 \\ &+ \sum_{k=1}^{\infty} \frac{1}{|B(x, r_k)|^2} \left(\int_{H_k(x)} |\nabla_A w(z_1, t)| dz_1 dt \right) \frac{h_k}{f(x_1 + r_{k+1})} \int_{H_k(x)} dy_1 dy_2, \end{aligned}$$

where for the last term we used that

$$\begin{aligned} |w_y(z_1, t)| &= \frac{f(z_1)}{f(z_1)} |w_y(z_1, t)| \leq \frac{1}{f(z_1)} |\nabla_A w(z_1, t)| \\ &\leq \frac{1}{f(x_1 + r_{k+1})} |\nabla_A w(z_1, t)| \quad \forall (z_1, z_2) \in E(x, r_k). \end{aligned}$$

Next, recall from Lemma 78 that $h_k \approx (r_k - r_{k+1}) \cdot f(x_1 + r_{k+1})$ by our choice of r_{k+1} in (7.1). Moreover, by the estimates above we have that $|H_k(x)| \approx |B(x, r_k)|$, and

$$\begin{aligned} |w(x)| &\lesssim \sum_{k=1}^{\infty} \frac{r_k - r_{k+1}}{|B(x, r_k)|} \left(\int_{H_k(x)} |\nabla_A w(s, y_2)| ds dy_2 \right) \\ (7.5) \quad &\lesssim \int_{\Gamma(x, r)} |\nabla_A w(y)| \left(\sum_{k=1}^{\infty} \frac{r_k - r_{k+1}}{|B(x, r_k)|} \mathbf{1}_{E(x, r_k)}(y) \right) dy. \end{aligned}$$

To make further estimates we need to consider two regions separately, namely;

case 1 $d(x, y) \geq \frac{1}{|F'(x_1)|}$. In this case we have

$$r_k > d(x, y) \geq \frac{1}{|F'(x_1)|},$$

which implies by Proposition 77 and (6.12)

$$r_k - r_{k+1} \approx \frac{1}{|F'(x_1 + r_k)|} \approx \frac{1}{|F'(x_1 + r_{k+2})|} < \frac{1}{|F'(x_1 + d(x, y))|}.$$

Therefore, we are left with

$$\begin{aligned} |w(x)| &\lesssim \int_{\Gamma(x, r)} |\nabla_A w(y)| \frac{1}{|F'(x_1 + d(x, y))|} \sum_{k: r_{k+1} < d(x, y) < r_k} \frac{1}{|B(x, r_k)|} dy \\ &\approx \int_{\Gamma(x, r)} |\nabla_A w(y)| \frac{1}{|F'(x_1 + d(x, y))|} \frac{1}{|B(x, d(x, y))|} dy. \end{aligned}$$

case 2 $d(x, y) < \frac{1}{|F'(x_1)|}$. We can write

$$\sum_{k: r_{k+1} < d(x, y) < r_k} \frac{r_k - r_{k+1}}{|B(x, r_k)|} \leq \sum_{k: r_{k+1} < d(x, y) < r_k} \frac{r_k}{|B(x, r_k)|} \lesssim \frac{d(x, y)}{|B(x, d(x, y))|},$$

which gives

$$|w(x)| \lesssim \int_{\Gamma(x, r)} |\nabla_A w(y)| \frac{d(x, y)}{|B(x, d(x, y))|}.$$

To finish the proof we need to compare the above estimates with $\hat{d}(x, y) \equiv \min \left\{ d(x, y), \frac{1}{|F'(x_1 + d(x, y))|} \right\}$. Since $|F'(x_1)|$ is a decreasing function of x_1 we have

$$d(x, y) \geq \frac{1}{|F'(x_1 + d(x, y))|} \implies d(x, y) \geq \frac{1}{|F'(x_1)|}$$

and therefore we are in **case 1** and have an estimate by $\hat{d}(x, y) = \frac{1}{|F'(x_1 + d(x, y))|}$. If the reverse inequality holds, namely,

$$d(x, y) < \frac{1}{|F'(x_1 + d(x, y))|}$$

we have to consider two subcases. First, if $d(x, y) \leq \frac{1}{|F'(x_1)|}$, then we are in **case 2** and have an estimate by $\hat{d}(x, y) = d(x, y)$. Finally, if

$$\frac{1}{|F'(x_1)|} \leq d(x, y) < \frac{1}{|F'(x_1 + d(x, y))|},$$

we are back in **case 1** but by Proposition 77 there holds

$$\frac{1}{|F'(x_1 + d(x, y))|} \approx d(x, y) - d(x, y)^* < d(x, y),$$

and again we have an estimate with $\hat{d}(x, y) = d(x, y)$. ■

As a simple corollary we obtain a connection between $\hat{d}(x, y)$ and the 'width' of the thickest part of a ball of radius $d(x, y)$, namely $d(x, y) - d^*(x, y)$, where if $r = d(x, y)$ then we write $r^* = d^*(x, y)$.

COROLLARY 85. *Let $d(x, y) > 0$ be the distance between any two points $x, y \in \Omega$ and let $d^*(x, y)$ be defined as in Section 3.4, and $\hat{d}(x, y)$ as defined in Lemma 84. Then*

$$\hat{d}(x, y) \approx d(x, y) - d^*(x, y)$$

PROOF. As before, we consider two cases

case 1 $d(x, y) \geq \frac{1}{|F'(x_1)|}$. In this case we have from Proposition 77

$$d(x, y) - d^*(x, y) \approx \frac{1}{|F'(x_1 + d(x, y))|}.$$

If $d(x, y) \geq \frac{1}{|F'(x_1 + d(x, y))|}$, then $\hat{d}(x, y) = \frac{1}{|F'(x_1 + d(x, y))|}$ and the claim is proved. If, on the other hand,

$$d(x, y) \leq \frac{1}{|F'(x_1 + d(x, y))|},$$

then $\hat{d}(x, y) = d(x, y)$ and

$$d(x, y) > d(x, y) - d^*(x, y) \approx \frac{1}{|F'(x_1 + d(x, y))|} \geq d(x, y),$$

and the claim follows.

case 2 $d(x, y) < \frac{1}{|F'(x_1)|}$. From Proposition 77 we have in this case

$$d(x, y) - d^*(x, y) \approx d(x, y).$$

From the monotonicity of the function $F'(x)$ we have

$$d(x, y) < \frac{1}{|F'(x_1)|} \leq \frac{1}{|F'(x_1 + d(x, y))|},$$

and therefore $\hat{d}(x, y) = d(x, y) \approx d(x, y) - d^*(x, y)$.

■

2. (1, 1) Sobolev and Poincaré inequalities

Define

$$(7.6) \quad K_r(x, y) \equiv \frac{\hat{d}(x, y)}{|B(x, d(x, y))|} \mathbf{1}_{\Gamma(x, r)}(y),$$

and for

$$y \in \Gamma(x, r) = \{y \in B(x, r) : x_1 \leq y_1 \leq x_1 + r, |y_2 - x_2| < h_{x, y}\},$$

let $h_{x, y} = h^*(x_1, y_1 - x_1)$. First, recall from Proposition 81 that we have an estimate

$$|B(x, d(x, y))| \approx h_{x, y}(d(x, y) - d^*(x, y))$$

and by Corollary 85 we have $|B(x, d(x, y))| \approx h_{x, y}\hat{d}(x, y)$. Thus,

$$K_r(x, y) \approx \frac{1}{h_{x, y}} \mathbf{1}_{\{(x, y) : x_1 \leq y_1 \leq x_1 + r, |y_2 - x_2| < h_{x, y}\}}(x, y).$$

Now denote the dual cone $\Gamma^*(y, r)$ by

$$\Gamma^*(y, r) \equiv \{x \in B(y, r) : y \in \Gamma(x, r)\}.$$

Then we have

$$(7.7) \quad \begin{aligned} \Gamma^*(y, r) &= \{x \in B(y, r) : x_1 \leq y_1 \leq x_1 + r, |y_2 - x_2| < h_{x, y}\} \\ &= \{x \in B(y, r) : y_1 - r \leq x_1 \leq y_1, |x_2 - y_2| < h_{x, y}\}, \end{aligned}$$

and consequently we get the ‘straight across’ estimate,

$$(7.8) \quad \int K_r(x, y) dx \approx \int_{y_1 - r}^{y_1} \left\{ \int_{y_2 - h_{x, y}}^{y_2 + h_{x, y}} \frac{1}{h_{x, y}} dx_2 \right\} dx_1 \approx \int_{x_1}^{x_1 + r} dy_1 = r.$$

As a result we obtain the following (1, 1) Sobolev inequality.

LEMMA 86. *For $w \in Lip_0(B(x_0, r))$ and ∇_A a degenerate gradient as above, we have*

$$\int_{B(x_0, r)} |w(x)| dx \leq Cr \int_{B(x_0, r)} |\nabla_A w(y)| dy.$$

PROOF. If $x \in B(x_0, r)$, then w satisfies the hypothesis of Lemma 84 in $B(x, 2(C+1)^2 r)$ for the constant C as in (7.2). Indeed, let r_k be defined by (7.1) for $x = y$ and $r_0 = 2(C+1)^2 r$, then

$$r_2 \geq \frac{1}{(C+1)^2} r_0 = 2r.$$

Hence, since

$$E(x, r_1) \equiv \{y : x_1 + r_2 \leq y_1 < x_1 + r_1, |y_2 - x_2| < h^*(x_1, z_1 - x_1)\},$$

we have that $E(x, r_1) \cap B(x_0, r) = \emptyset$ so $\int_{E(x, r_1)} w = 0$ so we may apply Lemma 84 in $B(x_0, 2((C+1)^2 + 1)r)$ for all $x \in B(x_0, r)$.

Let $R = 2 \left((C+1)^2 + 1 \right) r$. Using the subrepresentation inequality and (7.8) we have

$$\begin{aligned}
 \int |w(x)| \, dx &\leq \int \int_{\Gamma(x,R)} \frac{\widehat{d}(x,y)}{|B(x,d(x,y))|} |\nabla_A w(y)| \, dy dx \\
 &= \int \int K_R(x,y) |\nabla_A w(y)| \, dy dx \\
 &= \int \left\{ \int K_R(x,y) \, dx \right\} |\nabla_A w(y)| \, dy \\
 &\approx \int R |\nabla_A w(y)| \, dy \approx r \int |\nabla_A w(y)| \, dy.
 \end{aligned}$$

■

REMARK 87. The larger kernel $\widetilde{K}_r(x,y) \equiv \mathbf{1}_{\Gamma(x,r)}(y) \frac{d(x,y)}{|B(x,d(x,y))|}$, with \widehat{d} replaced by d , does **not** in general yield the (1, 1) Sobolev inequality. More precisely, the inequality

$$(7.9) \quad \int \int \widetilde{K}_r(x,y) |\nabla_A w(y)| \, dy dx \lesssim r \int |\nabla_A w(y)| \, dy, \quad 0 < r \ll 1,$$

fails in the case

$$F(x) = \frac{1}{x}, \quad x > 0.$$

To see this take $y_2 = 0$. We now make estimates on the integral

$$(7.10) \quad \int \widetilde{K}_r(x,y) \, dx \approx \int_{y_1-r}^{y_1} \left\{ \int_{y_2-h_{x,y}}^{y_2+h_{x,y}} \frac{1}{h_{x,y}} \frac{d(x,y)}{\widehat{d}(x,y)} dx_2 \right\} dx_1,$$

where $\widehat{d}(x,y) = \min \left\{ d(x,y), \frac{1}{|F'(x_1+d(x,y))|} \right\}$. Consider the region where

$$(7.11) \quad d(x,y) \geq \frac{1}{|F'(x_1+d(x,y))|} = (x_1+d(x,y))^2.$$

In this region we have

$$\frac{d(x,y)}{\widehat{d}(x,y)} = d(x,y) |F'(x_1+d(x,y))| = \frac{d(x,y)}{(x_1+d(x,y))^2}.$$

Moreover, since $d(x,y) \leq r$, we have

$$\frac{d(x,y)}{\widehat{d}(x,y)} \geq \frac{r}{(x_1+r)^2}.$$

On the other hand, we have $d(x,y) \geq y_1 - x_1$ and $d(x,y) \ll 1$, so the condition in (7.11) is guaranteed by $y_1 - x_1 \geq (x_1 + y_1 - x_1)^2$, i.e. $x_1 \leq y_1 - y_1^2$. We then have the following estimate for (7.10):

$$\int \widetilde{K}_r(x,y) dx \gtrsim \int_{y_1-r}^{y_1-y_1^2} \frac{r}{(x_1+r)^2} dx_1 = \frac{r(r-y_1^2)}{y_1(y_1-y_1^2+r)}.$$

Therefore, if $y_1 \leq r$, we have

$$\int \widetilde{K}_r(x,y) dx \gtrsim 1,$$

and (7.9) fails for small $r > 0$.

Now we turn to establishing the $(1, 1)$ *Poincaré* inequality. For this we will need the following extension of Lemma 79 in [RSaW]. Define the half metric ball

$$HB(0, r) = B(0, r) \cap \{(x, y) \in \mathbb{R}^2 : x > 0\}.$$

PROPOSITION 88. *Let the balls $B(0, r)$ and the degenerate gradient ∇_A be as above. There exists a constant C such that the Poincaré Inequality*

$$\iint_{HB(0, r)} |w - \bar{w}| dx dy \leq Cr \iint_{HB(0, r)} |\nabla_A w| dx dy$$

holds for any Lipschitz function w and sufficiently small $r > 0$. Here \bar{w} is the average defined by

$$\bar{w} = \frac{1}{|HB(0, r)|} \iint_{HB(0, r)} w dx dy.$$

2.1. Proof of Poincaré. The left hand side can be estimated by

$$\begin{aligned} \iint_{HB(0, r)} |w - \bar{w}| dx dy &= \iint_{HB(0, r)} \left| w(x_1, y_1) - \frac{1}{|HB(0, r)|} \iint_{HB(0, r)} w(x_2, y_2) dx_2 dy_2 \right| dx_1 dy_1 \\ &\leq \frac{1}{|HB(0, r)|} \int_{HB(0, r) \times HB(0, r)} |w(x_1, y_1) - w(x_2, y_2)| dx_1 dy_1 dx_2 dy_2 \end{aligned}$$

The idea now is to estimate the difference $|w(x_1, y_1) - w(x_2, y_2)|$ by the integral of ∇w along some path. Because the half metric ball is somewhat complicated geometrically, we can simplify the argument by applying the following lemma, sacrificing only the best constant C in the Poincaré inequality.

LEMMA 89. *Let (X, μ) be a measure space. If $\Omega \subset X$ is the disjoint union of 2 measurable subsets $\Omega = \Omega_1 \cup \Omega_2$ so that the measure of the subsets are comparable*

$$\frac{1}{C_1} \leq \frac{\mu(\Omega_1)}{\mu(\Omega_2)} \leq C_1$$

Then there exists a constant $C = C(C_1)$, such that

$$(7.12) \quad \iint_{\Omega \times \Omega} |w(x) - w(y)| d\mu(x) d\mu(y) \leq C \iint_{\Omega_1 \times \Omega_2} |w(x) - w(y)| d\mu(x) d\mu(y).$$

for any measurable function w defined on Ω .

PROOF. Define

$$S_{i,j} = \iint_{\Omega_i \times \Omega_j} |w(x) - w(y)| d\mu(x) d\mu(y), \quad i, j = 1, 2.$$

Since $\Omega = \Omega_1 \cup \Omega_2$, we can rewrite inequality (7.12) as

$$S_{1,1} + 2S_{1,2} + S_{2,2} \leq CS_{1,2}.$$

Now, we compute

$$\begin{aligned}
S_{1,1} &= \frac{1}{\mu(\Omega_2)} \iiint_{\Omega_1 \times \Omega_1 \times \Omega_2} |[w(x) - w(z)] + [w(z) - w(y)]| d\mu(x) d\mu(y) d\mu(z) \\
&\leq \frac{1}{\mu(\Omega_2)} \iiint_{\Omega_1 \times \Omega_1 \times \Omega_2} |w(x) - w(z)| d\mu(x) d\mu(y) d\mu(z) \\
&\quad + \frac{1}{\mu(\Omega_2)} \iiint_{\Omega_1 \times \Omega_1 \times \Omega_2} |w(y) - w(z)| d\mu(x) d\mu(y) d\mu(z) \\
&= \frac{2\mu(\Omega_1)}{\mu(\Omega_2)} \iint_{\Omega_1 \times \Omega_2} |w(x) - w(z)| d\mu(x) d\mu(z) = \frac{2\mu(\Omega_1)}{\mu(\Omega_2)} S_{1,2},
\end{aligned}$$

and similarly $S_{2,2} \leq \frac{2\mu(\Omega_2)}{\mu(\Omega_1)} S_{1,2}$. ■

We will apply this lemma with

$$\begin{aligned}
\Omega_1 &= B_+ = \{(x, y) \in HB(0, r_0) : r^* \leq x \leq r\}, \\
\Omega_2 &= B_- = \{(x, y) \in HB(0, r_0) : 0 \leq x \leq r^*\},
\end{aligned}$$

where r^* , B_+ and B_- are as in Lemma 83 above. Then from Lemma 83 we have

$$|\Omega_1| \approx |\Omega_2| \approx |B(0, r_0)|.$$

By Lemma 89, the proof of Proposition 88 reduces to the following inequality:

(7.13)

$$I = \iint_{\Omega_1 \times \Omega_2} |w(x_1, y_1) - w(x_2, y_2)| dx_1 dy_1 dx_2 dy_2 \leq C |HB(0, r_0)| r_0 \iint_{HB(0, r_0)} |\nabla_A w(x, y)| dx dy.$$

Let $P_1 = (x_1, y_1) \in \Omega_1$ and $P_2 = (x_2, y_2) \in \Omega_2$. We can connect P_1 and P_2 by first travelling vertically and then horizontally. This integral path is completely contained in the half metric ball. This immediately gives an inequality

$$|w(x_1, y_1) - w(x_2, y_2)| \leq \left| \int_{y_1}^{y_2} w_y(x_1, y) dy \right| + \left| \int_{x_1}^{x_2} w_x(x, y_2) dx \right|.$$

As a result, we have

$$\begin{aligned}
I &= \iint_{\Omega_1 \times \Omega_2} |w(x_1, y_1) - w(x_2, y_2)| dx_1 dy_1 dx_2 dy_2 \leq \iint_{\Omega_1 \times \Omega_2} \left| \int_{y_1}^{y_2} w_y(x_1, y) dy \right| dx_1 dy_1 dx_2 dy_2 \\
&\quad + \iint_{\Omega_1 \times \Omega_2} \left| \int_{x_1}^{x_2} w_x(x, y_2) dx \right| dx_1 dy_1 dx_2 dy_2 \\
&= I_1 + I_2
\end{aligned}$$

We first estimate the integral

$$I_1 = \iint_{\Omega_1 \times \Omega_2} |w(x_1, y_1) - w(x_2, y_2)| dx_1 dy_1 dx_2 dy_2 \leq \iint_{\Omega_1 \times \Omega_2} \left| \int_{y_1}^{y_2} w_y(x_1, y) dy \right| dx_1 dy_1 dx_2 dy_2$$

where $\Omega_1 = B_+$ and $\Omega_2 = B_-$. We have

$$\begin{aligned}
I_1 &\leq \int_{B_-} \int_{B_+} \int_{y_1}^{y_2} |w_y(x_1, y)| dy dx_1 dy_1 dx_2 dy_2 \leq \int_{B_-} \int_{B_+} \int_{y_1}^{y_2} \frac{1}{f(x_1)} |\nabla_A w(x_1, y)| dy dx_1 dy_1 dx_2 dy_2 \\
&\leq \int_{B_-} \int_{B_+} \frac{h(r)}{f(x_1)} |\nabla_A w(x_1, y)| dy dx_1 dx_2 dy_2,
\end{aligned}$$

where $h(r) \lesssim rf(r)$ is the “maximal height” given in Proposition 77. Moreover, for $x_1 \in B_+$ we have $|r - x_1| \leq 1/|F'(r)|$ and therefore $f(x_1) \approx f(r)$. This gives

$$\frac{h(r)}{f(x_1)} \leq r,$$

and substituting this into the above we get

$$I_1 \leq Cr|B_-| \int_{B_+} |\nabla_A w(x, y)| dx dy \leq Cr|B| \int_B |\nabla_A w(x, y)| dx dy.$$

To estimate

$$I_2 = \iint_{\Omega_1 \times \Omega_2} \left| \int_{x_1}^{x_2} w_x(x, y_2) dx \right| dx_1 dy_1 dx_2 dy_2,$$

we note that $|w_x(x, y_2)| \leq |\nabla_A w(x, y_2)|$, and therefore

$$\begin{aligned} I_2 &\leq \iint_{B_+ \times B_-} \left[\int_{(x, y_2) \in HB(0, r)} |\nabla_A w(x, y_2)| dx \right] dx_1 dy_1 dx_2 dy_2 \\ &\leq Cr|B_+| \int_B |\nabla_A w(x, y_2)| dx dy_2 \leq Cr|B| \int_B |\nabla_A w(x, y)| dx dy. \end{aligned}$$

This finishes the proof of inequality (7.13), and hence finishes the proof of the Poincaré inequality in Proposition 88.

3. Orlicz inequalities and submultiplicativity

Suppose that μ is a σ -finite measure on a set X , and $\Phi : [0, \infty) \rightarrow [0, \infty)$ is a Young function, which for our purposes is a convex piecewise differentiable (meaning there are at most finitely many points where the derivative of Φ may fail to exist, but right and left hand derivatives exist everywhere) function such that $\Phi(0) = 0$ and

$$\frac{\Phi(x)}{x} \rightarrow \infty \text{ as } x \rightarrow \infty.$$

Let L_*^Φ be the set of measurable functions $f : X \rightarrow \mathbb{R}$ such that the integral

$$\int_X \Phi(|f|) d\mu,$$

is finite, where as usual, functions that agree almost everywhere are identified. Since the set L_*^Φ may not be closed under scalar multiplication, we define L^Φ to be the linear span of L_*^Φ , and then define

$$\|f\|_{L^\Phi(\mu)} \equiv \inf \left\{ k \in (0, \infty) : \int_X \Phi\left(\frac{|f|}{k}\right) d\mu \leq 1 \right\}.$$

The Banach space $L^\Phi(\mu)$ is precisely the space of measurable functions f for which the norm $\|f\|_{L^\Phi(\mu)}$ is finite. The conjugate Young function $\tilde{\Phi}$ is defined by $\tilde{\Phi}' = (\Phi')^{-1}$ and can be used to give an equivalent norm

$$\|f\|_{L^\Phi(\mu)} \equiv \sup \left\{ \int_X |fg| d\mu : \int_X \tilde{\Phi}(|g|) d\mu \leq 1 \right\}.$$

However, in this paper, the homogeneity of the norm $\|f\|_{L^\Phi(\mu)}$ is not important, rather it is the iteration of Orlicz expressions that is critical. For this reason we will not need to invoke the classical

properties of these normed spaces, choosing instead to work directly with the nonhomogeneous expressions

$$\Phi^{(-1)} \left(\int_X \Phi(|f|) d\mu \right).$$

In our setting of infinitely degenerate metrics in the plane, the metrics we consider are elliptic away from the x_2 axis, and are invariant under vertical translations. As a consequence, we need only consider Sobolev inequalities for the metric balls $B(0, r)$ centered at the origin. So from now on we consider $X = \mathbb{R}^2$ and the metric balls $B(0, r)$ associated to one of the geometries F considered in Part 2.

First we recall that the optimal form of the degenerate Orlicz-Sobolev *norm* inequality for balls is

$$\|w\|_{L^\Psi(\mu_{r_0})} \leq C r_0 \|\nabla_A w\|_{L^\Omega(\mu_{r_0})},$$

where $d\mu_{r_0}(x) = \frac{dx}{|B(0, r_0)|}$, the balls $B(0, r_0)$ are control balls for a metric A , and the Young function Ψ is a ‘bump up’ of the Young function Ω . We will instead obtain the nonhomogeneous form of this inequality where $L^\Omega(\mu_{r_0}) = L^1(\mu_{r_0})$ is the usual Lebesgue space, and the factor r_0 on the right hand side is replaced by a suitable superradius $\varphi(r_0)$, namely

$$(7.14) \quad \Phi^{(-1)} \left(\int_{B(0, r_0)} \Phi(w) d\mu_{r_0} \right) \leq C \varphi(r_0) \|\nabla_A w\|_{L^1(\mu_{r_0})}, \quad w \in Lip_0(X),$$

which we refer to as the (Φ, φ) -Sobolev Orlicz bump inequality. In fact, with the positive operator $T_{B(0, r_0)} : L^1(\mu_{r_0}) \rightarrow L^\Phi(\mu_{r_0})$ defined by

$$T_{B(0, r_0)} g(x) \equiv \int_{B(0, r_0)} K_{B(0, r_0)}(x, y) g(y) dy$$

with kernel $K_{B(0, r_0)}$ defined as in (7.6), we will obtain the following stronger inequality,

$$(7.15) \quad \Phi^{(-1)} \left(\int_{B(0, r_0)} \Phi(T_{B(0, r_0)} g) d\mu_{r_0} \right) \leq C \varphi(r_0) \|g\|_{L^1(\mu_{r_0})},$$

which we refer to as the *strong* (Φ, φ) -Sobolev Orlicz bump inequality, and which is stronger by the subrepresentation inequality $w \lesssim T_{B(0, r_0)} \nabla_A w$ on $B(0, r_0)$. But this inequality cannot in general be reversed. When we wish to emphasize that we are working with (7.14), we will often call it the *standard* (Φ, φ) -Sobolev Orlicz bump inequality.

3.1. Submultiplicative extensions. In our application to Moser iteration the convex bump function $\Phi(t)$ is assumed to satisfy in addition:

- The function $\frac{\Phi(t)}{t}$ is positive, nondecreasing and tends to ∞ as $t \rightarrow \infty$;
- Φ is submultiplicative on an interval (E, ∞) for some $E > 1$:

$$(7.16) \quad \Phi(ab) \leq \Phi(a) \Phi(b), \quad a, b > E.$$

Note that if we consider more generally the quasi-submultiplicative condition,

$$(7.17) \quad \Phi(ab) \leq K \Phi(a) \Phi(b), \quad a, b > E,$$

for some constant K , then $\Phi(t)$ satisfies (7.17) if and only if $\Phi_K(t) \equiv K \Phi(t)$ satisfies (7.16). Thus we can always rescale a quasi-submultiplicative function to be submultiplicative.

Now let us consider the *linear extension* of Φ defined on $[E, \infty)$ to the entire positive real axis $(0, \infty)$ defined by

$$\Phi(t) = \frac{\Phi(E)}{E}t, \quad 0 \leq t \leq E.$$

We claim that this extension of Φ is submultiplicative on $(0, \infty)$, i.e.

$$\Phi(ab) \leq \Phi(a)\Phi(b), \quad a, b > 0.$$

In fact, the identity $\Phi(t)/t = \Phi(\max\{t, E\})/\max\{t, E\}$ and the monotonicity of $\Phi(t)/t$ imply

$$\frac{\Phi(ab)}{ab} \leq \frac{\Phi(\max\{a, E\})\Phi(\max\{b, E\})}{\max\{a, E\}\max\{b, E\}} \leq \frac{\Phi(\max\{a, E\})}{\max\{a, E\}} \cdot \frac{\Phi(\max\{b, E\})}{\max\{b, E\}} = \frac{\Phi(a)}{a} \frac{\Phi(b)}{b}.$$

CONCLUSION 90. *If $\Phi : [E, \infty) \rightarrow \mathbb{R}^+$ is a submultiplicative piecewise differentiable convex function so that $\Phi(t)/t$ is nondecreasing, then we can extend Φ to a submultiplicative piecewise differentiable convex function on $[0, \infty)$ that vanishes at 0 if and only if*

$$(7.18) \quad \Phi'(E) \geq \frac{\Phi(E)}{E}.$$

3.1.1. *An explicit family of Orlicz bumps.* We now consider the *near power bump* case $\Phi_m(t) = e^{((\ln t)^{\frac{1}{m}} + 1)^m}$ for $m > 1$. In the special case that $m > 1$ is an integer we can expand the m^{th} power in

$$\ln \Phi_m(e^s) = \left(s^{\frac{1}{m}} + 1\right)^m = \sum_{k=0}^m \binom{m}{k} s^{\frac{k}{m}},$$

and using the inequality $1 \leq \left(\frac{s}{s+t}\right)^\alpha + \left(\frac{t}{s+t}\right)^\alpha$ for $s, t > 0$ and $0 \leq \alpha \leq 1$, we see that $\Theta_m(s) \equiv \ln \Phi_m(e^s)$ is subadditive on $(0, \infty)$, hence Φ_m is submultiplicative on $(1, \infty)$. In fact, it is not hard to see that for $m > 1$, $\Theta_m(s) = \left(s^{\frac{1}{m}} + 1\right)^m$ is subadditive on $(0, \infty)$, and so Φ_m is submultiplicative on $(1, \infty)$.

We now compute that for any $t > 0$ we have

$$\begin{aligned} \Phi'_m(t) &= \Phi_m(t) m \left((\ln t)^{\frac{1}{m}} + 1 \right)^{m-1} \frac{1}{m} (\ln t)^{\frac{1}{m}-1} \frac{1}{t} \\ &= \frac{\Phi_m(t)}{t} \left(1 + \frac{1}{(\ln t)^{\frac{1}{m}}} \right)^{m-1}, \end{aligned}$$

and so for $E > 1$ we have

$$\Phi'_m(E) = \frac{\Phi_m(E)}{E} \left(1 + \frac{1}{(\ln E)^{\frac{1}{m}}} \right)^{m-1} > \frac{\Phi_m(E)}{E}.$$

Moreover, we compute

$$\begin{aligned} \Phi''_m(t) &= \frac{\Phi_m(t)}{t^2} \left(1 + (\ln t)^{-\frac{1}{m}} \right)^{2m-2} - \frac{\Phi_m(t)}{t^2} \left(1 + (\ln t)^{-\frac{1}{m}} \right)^{m-1} \\ &\quad - \frac{m-1}{m} \frac{\Phi_m(t)}{t^2} \left(1 + (\ln t)^{-\frac{1}{m}} \right)^{m-2} (\ln t)^{-\frac{1}{m}-1} \\ &= \frac{\Phi_m(t)}{t^2} \left(1 + (\ln t)^{-\frac{1}{m}} \right)^{m-2} F_m(t), \end{aligned}$$

where

$$\begin{aligned} F_m(t) &= \left(1 + (\ln t)^{-\frac{1}{m}}\right)^m - \frac{m-1}{m} (\ln t)^{-\frac{m+1}{m}} - 1 - (\ln t)^{-\frac{1}{m}}; \\ F_m(e) &= 2^m - \frac{m-1}{m} - 2 > 0, \end{aligned}$$

for $m > 1$. This shows that Φ_m is convex on (e, ∞) , and so by Conclusion 90 we can extend Φ_m to a positive increasing submultiplicative convex function on $[0, \infty)$. However, due to technical calculations below, it is convenient to take $E = E_m = e^{2^m}$, and as a consequence we will work from now on with the definition

$$(7.19) \quad \Phi_m(t) \equiv \begin{cases} e^{((\ln t)^{\frac{1}{m}+1})^m} & \text{if } t \geq E = E_m = e^{2^m} \\ \frac{\Phi_m(E)}{E} t & \text{if } 0 \leq t \leq E = E_m = e^{2^m} \end{cases},$$

where $m > 1$ will be explicitly mentioned or understood from the context. Later, for use in establishing continuity of weak solutions, we will introduce a positive increasing convex function $\Psi(t)$ that is essentially $\Phi_m^{(-1)}$ for small t and affine for large t . This function will turn out to be quasi-supermultiplicative.

4. Sobolev inequalities for submultiplicative bumps when $t > M$

Recall the operator $T_{B(0,r_0)} : L^1(\mu_{r_0}) \rightarrow L^\Phi(\mu_{r_0})$ defined by

$$T_{B(0,r_0)}g(x) \equiv \int_{B(0,r_0)} K_{B(0,r_0)}(x,y) g(y) dy$$

with kernel K defined as in (7.6). We begin by proving that the bound (7.15) holds if the following endpoint inequality holds:

$$(7.20) \quad \Phi^{-1} \left(\sup_{y \in B} \int_B \Phi(K(x,y)|B|\alpha) d\mu(x) \right) \leq C\alpha\varphi(r).$$

for all $\alpha > 0$. Indeed, if (7.20) holds, then with $g = |\nabla_A w|$ and $\alpha = \|g\|_{L^1} = \|\nabla_A w\|_{L^1}$, we have using first the subrepresentation inequality, and then Jensen's inequality applied to the convex function Φ ,

$$\begin{aligned} \int_B \Phi(w) d\mu(x) &\lesssim \int_B \Phi \left(\int_B K(x,y) |B| \|g\|_{L^1(\mu)} \frac{g(y) d\mu(y)}{\|g\|_{L^1(\mu)}} \right) d\mu(x) \\ &\leq \int_B \int_B \Phi(K(x,y) |B| \|g\|_{L^1(\mu)}) \frac{g(y) d\mu(y)}{\|g\|_{L^1(\mu)}} d\mu(x) \\ &\leq \int_B \left\{ \sup_{y \in B} \int_B \Phi(K(x,y) |B| \|g\|_{L^1(\mu)}) d\mu(x) \right\} \frac{g(y) d\mu(y)}{\|g\|_{L^1(\mu)}} \\ &\leq \Phi(C\varphi(r) \|g\|_{L^1(\mu)}) \int_B \frac{g(y) d\mu(y)}{\|g\|_{L^1(\mu)}} = \Phi(C\varphi(r) \|g\|_{L^1(\mu)}), \end{aligned}$$

and so

$$\Phi^{-1} \left(\int_B \Phi(w) d\mu(x) \right) \lesssim C\varphi(r) \|g\|_{L^1(\mu)}.$$

The converse follows from Fatou's lemma, but we will not need this. Note that (7.20) is obtained from (7.15) by replacing $g(y) dy$ with the point mass $|B|\alpha\delta_x(y)$ so that $Tg(x) \rightarrow K(x,y) |B| \alpha$.

REMARK 91. *The inhomogeneous condition (7.20) is in general stronger than its homogeneous counterpart*

$$\sup_{y \in B(0, r_0)} \|K_{B(0, r_0)}(\cdot, y) |B(0, r_0)|\|_{L^\Phi(\mu_{r_0})} \leq C\varphi(r_0),$$

but is equivalent to it when Φ is submultiplicative. We will not however use this observation.

Now we turn to the explicit near power bumps Φ in (7.19), which satisfy

$$\Phi(t) = \Phi_m(t) = e^{((\ln t)^{\frac{1}{m}+1})^m}, \quad t > e^{2^m},$$

for $m \in (1, \infty)$. Let $\psi(t) = \left(1 + (\ln t)^{-\frac{1}{m}}\right)^m - 1$ for $t > E = e^{2^m}$ and write $\Phi(t) = t^{1+\psi(t)}$.

PROPOSITION 92. *Let $0 < r_0 < 1$ and $C_m > 0$. Suppose that the geometry F satisfies the monotonicity property:*

$$(7.21) \quad \varphi(r) \equiv \frac{1}{|F'(r)|} e^{C_m \left(\frac{|F'(r)|^2}{F''(r)} + 1 \right)^{m-1}} \quad \text{is an increasing function of } r \in (0, r_0).$$

Then the (Φ, φ) -Sobolev inequality (7.15) holds with geometry F , with φ as in (7.21) and with Φ as in (7.19), $m > 1$.

For fixed $\Phi = \Phi_m$ with $m > 1$, we now consider the geometry of balls defined by

$$\begin{aligned} F_{k,\sigma}(r) &= \left(\ln \frac{1}{r} \right) \left(\ln^{(k)} \frac{1}{r} \right)^\sigma; \\ f_{k,\sigma}(r) &= e^{-F_{k,\sigma}(r)} = e^{-(\ln \frac{1}{r})(\ln^{(k)} \frac{1}{r})^\sigma}, \end{aligned}$$

where $k \in \mathbb{N}$ and $\sigma > 0$.

COROLLARY 93. *The strong (Φ, φ) -Sobolev inequality (7.15) with $\Phi = \Phi_m$ as in (7.19), $m > 1$, and geometry $F = F_{k,\sigma}$ holds if*
(either) $k \geq 2$ and $\sigma > 0$ and $\varphi(r_0)$ is given by

$$\varphi(r_0) = r_0^{\frac{1-C_m \left(\ln^{(k)} \frac{1}{r_0} \right)^\sigma}{\ln \frac{1}{r_0}}}, \quad \text{for } 0 < r_0 \leq \beta_{m,\sigma},$$

for positive constants C_m and $\beta_{m,\sigma}$ depending only on m and σ ;

(or) $k = 1$ and $\sigma < \frac{1}{m-1}$ and $\varphi(r_0)$ is given by

$$\varphi(r_0) = r_0^{\frac{1-C_m}{\left(\ln \frac{1}{r_0} \right)^{1-\sigma(m-1)}}}, \quad \text{for } 0 < r_0 \leq \beta_{m,\sigma},$$

for positive constants C_m and $\beta_{m,\sigma}$ depending only on m and σ .

Conversely, the standard (Φ, φ) -Sobolev inequality (7.14) with Φ as in (7.19), $m > 1$, fails if $k = 1$ and $\sigma > \frac{1}{m-1}$.

PROOF OF PROPOSITION 92. It suffices to prove the endpoint inequality (7.20). However, since the estimates we use on the kernel $K(x, y)$ are essentially symmetric in x and y , see e.g. the formula (7.7) for the dual cone Γ^* , we will instead prove the ‘dual’ of (7.20) in which x and y are interchanged:

$$(7.22) \quad \Phi^{-1} \left(\sup_{x \in B} \int_B \Phi(K(x, y) |B| \alpha) d\mu(y) \right) \leq C\alpha \varphi(r(B)), \quad \alpha > 0,$$

for the balls and kernel associated with our geometry F , the Orlicz bump Φ , and the function $\varphi(r)$ satisfying (7.21). Fix parameters $m > 1$ and $t_m > 1$. Now we consider the specific function $\omega(r(B))$ given by

$$\omega(r(B)) = \frac{1}{t_m |F'(r(B))|}.$$

Using the submultiplicativity of Φ we have

$$\begin{aligned} \int_B \Phi(K(x, y)|B|\alpha) d\mu(y) &= \int_B \Phi\left(\frac{K(x, y)|B|}{\omega(r(B))} \alpha \omega(r(B))\right) d\mu(y) \\ &\leq \Phi(\alpha \omega(r(B))) \int_B \Phi\left(\frac{K(x, y)|B|}{\omega(r(B))}\right) d\mu(y) \end{aligned}$$

and we will now prove

$$(7.23) \quad \int_B \Phi\left(\frac{K(x, y)|B|}{\omega(r(B))}\right) d\mu(y) \leq C_m \varphi(r(B)) |F'(r(B))|,$$

for all small balls B of radius $r(B)$ centered at the origin. Altogether this will give us

$$\int_B \Phi(K(x, y)|B|\alpha) d\mu(y) \leq C_m \varphi(r(B)) |F'(r(B))| \Phi\left(\frac{\alpha}{t_m |F'(r(B))|}\right).$$

Now we note that $x\Phi(y) = xy \frac{\Phi(y)}{y} \leq xy \frac{\Phi(xy)}{xy} = \Phi(xy)$ for $x \geq 1$ since $\frac{\Phi(t)}{t}$ is monotone increasing.

But from (7.21) we have $\varphi(r) |F'(r)| = e^{C_m \left(\frac{|F'(r)|^2}{F''(r)} + 1\right)^{m-1}} \gg 1$ and so

$$\int_B \Phi(K(x, y)|B|\alpha) d\mu(y) \leq \Phi\left(C_m \varphi(r(B)) |F'(r(B))| \alpha \frac{1}{t_m |F'(r(B))|}\right) = \Phi\left(\frac{C_m}{t_m} \alpha \varphi(r(B))\right),$$

which is (7.22) with $C = \frac{C_m}{t_m}$. Thus it remains to prove (7.23).

So we now take $B = B(0, r_0)$ with $r_0 \ll 1$ so that $\omega(r(B)) = \omega(r_0)$. First, recall

$$|B(0, r_0)| \approx \frac{f(r_0)}{|F'(r_0)|^2},$$

and

$$K(x, y) \approx \frac{1}{h_{y_1 - x_1}} \approx \begin{cases} \frac{1}{rf(x_1)}, & 0 < r = y_1 - x_1 < \frac{1}{|F'(x_1)|} \\ \frac{|F'(x_1 + r)|}{f(x_1 + r)}, & 0 < r = y_1 - x_1 \geq \frac{1}{|F'(x_1)|} \end{cases}.$$

Next, write $\Phi(t)$ as

$$(7.24) \quad \Phi(t) = t^{1+\psi(t)}, \quad \text{for } t > 0,$$

where for $t \geq E$,

$$\begin{aligned} t^{1+\psi(t)} &= \Phi(t) = e^{((\ln t)^{\frac{1}{m}} + 1)^m} = t^{(1 + (\ln t)^{-\frac{1}{m}})^m} \\ \implies \psi(t) &= \left(1 + (\ln t)^{-\frac{1}{m}}\right)^m - 1 \approx \frac{m}{(\ln t)^{1/m}}, \end{aligned}$$

and for $t < E$,

$$\begin{aligned} t^{1+\psi(t)} &= \Phi(t) = \frac{\Phi(E)}{E} t \\ \implies (1 + \psi(t)) \ln t &= \ln \frac{\Phi(E)}{E} + \ln t \\ \implies \psi(t) &= \frac{\ln \frac{\Phi(E)}{E}}{\ln t}. \end{aligned}$$

Now temporarily fix $x = (x_1, x_2) \in B_+(0, r_0) \equiv \{x \in B(0, r_0) : x_1 > 0\}$. We then have for $-x_1 < a < b < r_0 - x_1$ that

$$\begin{aligned} \mathcal{I}_{a,b}(x) &\equiv \int_{\{y \in B_+(0, r_0) : a \leq y_1 - x_1 \leq b\}} \Phi \left(K_{B(0, r_0)}(x, y) \frac{|B(0, r_0)|}{\omega(r_0)} \right) \frac{dy}{|B(0, r_0)|} \\ &= \int_{a+x_1}^{b+x_1} \left\{ \int_{x_2-h_{y_1-x_1}}^{x_2+h_{y_1-x_1}} \Phi \left(\frac{1}{h_{y_1-x_1}} |B(0, r_0)| \frac{|B(0, r_0)|}{\omega(r_0)} \right) dy_2 \right\} \frac{dy_1}{|B(0, r_0)|} \\ &= \int_{a+x_1}^{b+x_1} 2h_{y_1-x_1} \Phi \left(\frac{1}{h_{y_1-x_1}} \frac{|B(0, r_0)|}{\omega(r_0)} \right) \frac{dy_1}{|B(0, r_0)|} \\ &= \int_{a+x_1}^{b+x_1} 2h_{y_1-x_1} \left(\frac{1}{h_{y_1-x_1}} \frac{|B(0, r_0)|}{\omega(r_0)} \right) \left(\frac{1}{h_{y_1-x_1}} \frac{|B(0, r_0)|}{\omega(r_0)} \right)^{\psi \left(\frac{1}{h_{y_1-x_1}} \frac{|B(0, r_0)|}{\omega(r_0)} \right)} \frac{dy_1}{|B(0, r_0)|} \end{aligned}$$

which simplifies to

$$\begin{aligned} \mathcal{I}_{a,b}(x) &= \frac{2}{\omega(r_0)} \int_{a+x_1}^{b+x_1} \left(\frac{1}{h_{y_1-x_1}} \frac{|B(0, r_0)|}{\omega(r_0)} \right)^{\psi \left(\frac{1}{h_{y_1-x_1}} \frac{|B(0, r_0)|}{\omega(r_0)} \right)} dy_1 \\ &= \frac{2}{\omega(r_0)} \int_a^b \left(\frac{1}{h_r} \frac{|B(0, r_0)|}{\omega(r_0)} \right)^{\psi \left(\frac{1}{h_r} \frac{|B(0, r_0)|}{\omega(r_0)} \right)} dr. \end{aligned}$$

Thus we have

$$\begin{aligned} &\int_{B_+(0, r_0)} \Phi \left(K_{B(0, r_0)}(x, y) \frac{|B(0, r_0)|}{\omega(r_0)} \right) \frac{dy}{|B(0, r_0)|} \\ &= \mathcal{I}_{-x_1, r_0-x_1}(x) \\ &= \frac{2}{\omega(r_0)} \int_{-x_1}^{r_0-x_1} \left(\frac{1}{h_r} \frac{|B(0, r_0)|}{\omega(r_0)} \right)^{\psi \left(\frac{1}{h_r} \frac{|B(0, r_0)|}{\omega(r_0)} \right)} dr. \end{aligned}$$

To prove (7.23) it suffices to obtain the following estimate for the integral \mathcal{I}_{0, r_0-x_1} , since the complementary integral $\mathcal{I}_{-x_1, 0}$ can be handled similarly to obtain the same estimate:

$$(7.25) \quad \mathcal{I}_{0, r_0-x_1} = \frac{1}{\omega(r_0)} \int_0^{r_0-x_1} \left(\frac{|B(0, r_0)|}{h_r \omega(r_0)} \right)^{\psi \left(\frac{|B(0, r_0)|}{h_r \omega(r_0)} \right)} dr \leq C_m \varphi(r_0) |F'(r_0)|,$$

where C_0 is a sufficiently large positive constant.

To prove this we divide the interval $(0, r_0 - x_1)$ of integration in r into three regions:

- (1): the small region \mathcal{S} where $\frac{|B(0, r_0)|}{h_r \omega(r_0)} \leq E$,
- (2): the big region \mathcal{R}_1 that is disjoint from \mathcal{S} and where $r = y_1 - x_1 < \frac{1}{|F'(x_1)|}$ and
- (3): the big region \mathcal{R}_2 that is disjoint from \mathcal{S} and where $r = y_1 - x_1 \geq \frac{1}{|F'(x_1)|}$.

In the small region \mathcal{S} we use that Φ is linear on $[0, E]$ to obtain that the integral in the right hand side of (7.25), when restricted to those $r \in (0, r_0 - x_1)$ for which $\frac{|B(0, r_0)|}{h_r \omega(r_0)} \leq E$, is equal to

$$\begin{aligned} & \frac{1}{\omega(r_0)} \int_0^{r_0 - x_1} \left(\frac{|B(0, r_0)|}{h_r \omega(r_0)} \right)^{\frac{\ln \frac{\Phi(E)}{E}}{\ln \left(\frac{|B(0, r_0)|}{h_r \omega(r_0)} \right)}} dr \\ &= \frac{1}{\omega(r_0)} \int_0^{r_0 - x_1} e^{\ln \frac{\Phi(E)}{E}} dr = \frac{1}{\omega(r_0)} \frac{\Phi(E)}{E} (r_0 - x_1) \\ &\leq \frac{\Phi(E)}{E} t_m r_0 |F'(r_0)|, \end{aligned}$$

since $\omega(r_0) = \frac{1}{t_m |F'(r_0)|}$.

We now turn to the first big region \mathcal{R}_1 where we have $h_{y_1 - x_1} \approx r f(x_1)$. The condition that \mathcal{R}_1 is disjoint from \mathcal{S} gives

$$\begin{aligned} \frac{|B(0, r_0)|}{r f(x_1) \omega(r_0)} &> E, \quad \text{i.e. } r < \frac{A}{E}; \\ \text{where } A &= A(x_1) \equiv \frac{|B(0, r_0)|}{f(x_1) \omega(r_0)}, \end{aligned}$$

and so

$$\begin{aligned} & \int_{\mathcal{R}_1} \Phi \left(K_{B(0, r_0)}(x, y) \frac{|B(0, r_0)|}{\omega(r_0)} \right) \frac{dy}{|B(0, r_0)|} \\ &= \mathcal{I}_{0, \min\left\{\frac{A}{E}, \frac{1}{|F'(x_1)|}\right\}}(x) \\ &= \frac{1}{\omega(r_0)} \int_0^{\min\left\{\frac{A}{E}, \frac{1}{|F'(x_1)|}\right\}} \left(\frac{|B(0, r_0)|}{h_r \omega(r_0)} \right)^{\psi\left(\frac{|B(0, r_0)|}{h_r \omega(r_0)}\right)} dr \\ &= \frac{1}{\omega(r_0)} \int_0^{\min\left\{\frac{A}{E}, \frac{1}{|F'(x_1)|}\right\}} \left(\frac{A}{r} \right)^{\psi\left(\frac{A}{r}\right)} dr. \end{aligned}$$

We claim that

$$(7.26) \quad \frac{1}{\omega(r_0)} \int_0^{\min\left\{\frac{A}{E}, \frac{1}{|F'(x_1)|}\right\}} \left(\frac{A}{r} \right)^{\psi\left(\frac{A}{r}\right)} dr \lesssim \Phi(t_m),$$

where we recall

$$\begin{aligned} A &= A(x_1) \equiv \frac{f(r_0)}{f(x_1) |F'(r_0)|^2 \omega(r_0)} = \frac{c}{f(x_1)} \\ \text{and } c &= c(r_0) \equiv \frac{f(r_0)}{\omega(r_0) |F'(r_0)|^2} = \frac{t_m f(r_0)}{|F'(r_0)|}. \end{aligned}$$

Now if $\frac{A}{E} \leq \frac{1}{|F'(x_1)|}$, then

$$\begin{aligned}
& \frac{1}{\omega(r_0)} \int_0^{\min\left\{\frac{A}{E}, \frac{1}{|F'(x_1)|}\right\}} \left(\frac{A}{r}\right)^{\psi\left(\frac{A}{r}\right)} dr \\
&= \frac{1}{\omega(r_0)} \int_0^{\frac{A}{E}} \left(\frac{A}{r}\right)^{\psi\left(\frac{A}{r}\right)} dr = \frac{1}{\omega(r_0)} A \int_E^\infty t^{\psi(t)} \frac{dt}{t^2} \\
&\leq \frac{1}{\omega(r_0)} A C_\varepsilon \int_{E_m}^\infty t^{\varepsilon-2} dt = C_{\varepsilon,m} \frac{1}{\omega(r_0)} A \leq \frac{1}{\omega(r_0)} \frac{C_{\varepsilon,m} M}{|F'(x_1)|} \\
&\leq \frac{1}{\omega(r_0)} \frac{C_{\varepsilon,m} M}{|F'(r_0)|} \leq C_{\varepsilon,m} M \frac{r_0}{\omega(r_0)} = C_{\varepsilon,m} M t_m r_0 |F'(r_0)|,
\end{aligned}$$

which proves (7.25) if $\frac{A}{E} \leq \frac{1}{|F'(x_1)|}$ since $r_0 \leq \varphi(r_0)$.

So we now suppose that $\frac{A}{E} > \frac{1}{|F'(x_1)|}$. Making a change of variables

$$R = \frac{A}{r} = \frac{A(x_1)}{r},$$

we obtain

$$\frac{1}{\omega(r_0)} \int_0^{\frac{1}{|F'(x_1)|}} \left(\frac{A}{r}\right)^{\psi\left(\frac{A}{r}\right)} dr = \frac{1}{\omega(r_0)} A \int_{A|F'(x_1)|}^\infty R^{\psi(R)-2} dR.$$

Integrating by parts gives

$$\begin{aligned}
\int_{A|F'(x_1)|}^\infty R^{\psi(R)-2} dR &= \int_{A|F'(x_1)|}^\infty R^{\psi(R)+1} \left(-\frac{1}{2R^2}\right)' dR \\
&= -\frac{R^{\psi(R)+1}}{2R^2} \Big|_{A|F'(x_1)|}^\infty + \int_{A|F'(x_1)|}^\infty \left(R^{\psi(R)+1}\right)' \frac{1}{2R^2} dR \\
&\leq \frac{(A|F'(x_1)|)^{\psi(A|F'(x_1)|)}}{2A|F'(x_1)|} + \int_{A|F'(x_1)|}^\infty \frac{1}{2} R^{\psi(R)-2} \left(1 + C \frac{m-1}{(\ln R)^{\frac{1}{m}}}\right) dR \\
&\leq \frac{(A|F'(x_1)|)^{\psi(A|F'(x_1)|)}}{2A|F'(x_1)|} + \frac{1 + C \frac{m-1}{(\ln E)^{\frac{1}{m}}}}{2} \int_{A|F'(x_1)|}^\infty R^{\psi(R)-2} dR,
\end{aligned}$$

where we used

$$|\psi'(R)| \leq C \frac{1}{R} \frac{1}{(\ln R)^{\frac{m+1}{m}}}.$$

Taking E large enough depending on m we can assure

$$\frac{1 + C \frac{m-1}{(\ln E)^{\frac{1}{m}}}}{2} \leq \frac{3}{4},$$

which gives

$$\int_{A|F'(x_1)|}^\infty R^{\psi(R)-2} dR \lesssim \frac{(A|F'(x_1)|)^{\psi(A|F'(x_1)|)}}{A|F'(x_1)|},$$

and therefore

$$\begin{aligned} \mathcal{I}_{0, \frac{1}{|F'(x_1)|}}(x) &= \frac{1}{\omega(r_0)} A \int_{A|F'(x_1)|}^{\infty} R^{\psi(R)-2} dR \\ &\lesssim \frac{1}{\omega(r_0) |F'(x_1)|} (A(x_1) |F'(x_1)|)^{\psi(A(x_1)|F'(x_1)|)}; \\ c &= f(x_1) A(x_1) = \frac{f(r_0)}{\omega(r_0) |F'(r_0)|^2} = \frac{t_m f(r_0)}{|F'(r_0)|}, \end{aligned}$$

where we recall that we have assumed the condition

$$(7.27) \quad A(x_1) |F'(x_1)| = \frac{f(r_0)}{f(x_1) |F'(r_0)|^2 \omega(r_0)} |F'(x_1)| = c \frac{|F'(x_1)|}{f(x_1)} \geq E.$$

We now look for the maximum of the function on the right hand side

$$\begin{aligned} \mathcal{F}(x_1) &\equiv \frac{1}{\omega(r_0) |F'(x_1)|} (A(x_1) |F'(x_1)|)^{\psi(A(x_1)|F'(x_1)|)} \\ &= t_m |F'(r_0)| \frac{1}{|F'(x_1)|} \left(c(r_0) \frac{|F'(x_1)|}{f(x_1)} \right)^{\psi\left(c(r_0) \frac{|F'(x_1)|}{f(x_1)}\right)} \end{aligned}$$

where

$$c(r_0) = \frac{t_m f(r_0)}{|F'(r_0)|}.$$

Using the definition of $\psi(t)$ and $B(x_1) \equiv \ln \left[c(r_0) \frac{|F'(x_1)|}{f(x_1)} \right]$, we can rewrite $\mathcal{F}(x_1)$ as

$$(7.28) \quad \mathcal{F}(x_1) = t_m |F'(r_0)| \frac{1}{|F'(x_1)|} \exp \left(\left(1 + B(x_1)^{\frac{1}{m}} \right)^m - B(x_1) \right).$$

Let $x_1^* \in (0, r_0]$ be the point at which \mathcal{F} takes its maximum. Differentiating $\mathcal{F}(x_1)$ with respect to x_1 and then setting the derivative equal to zero, we obtain that x_1^* satisfies the equation,

$$\frac{F''(x_1^*)}{|F'(x_1^*)|^2} = \left(\left(1 + B(x_1^*)^{-\frac{1}{m}} \right)^{m-1} - 1 \right) \left(1 + \frac{F''(x_1^*)}{|F'(x_1^*)|^2} \right).$$

Simplifying gives the following implicit expression for x_1^* that maximizes $\mathcal{F}(x_1)$

$$B(x_1^*) = \ln \left[c(r_0) \frac{|F'(x_1^*)|}{f(x_1^*)} \right] = \left(\left(1 + \frac{F''(x_1^*)}{|F'(x_1^*)|^2 + F''(x_1^*)} \right)^{\frac{1}{m-1}} - 1 \right)^{-m}.$$

To estimate $\mathcal{F}(x_1^*)$ in an effective way, we set $b(x_1^*) \equiv \frac{F''(x_1^*)}{|F'(x_1^*)|^2 + F''(x_1^*)}$ and begin with

$$\begin{aligned} \left(1 + B(x_1)^{\frac{1}{m}} \right)^m - B(x_1) &= \left(1 + \left(\ln \left[c(r_0) \frac{|F'(x_1^*)|}{f(x_1^*)} \right] \right)^{\frac{1}{m}} \right)^m - \ln \left[c(r_0) \frac{|F'(x_1^*)|}{f(x_1^*)} \right] \\ &= \frac{\left(1 + \frac{F''(x_1^*)}{|F'(x_1^*)|^2 + F''(x_1^*)} \right)^{\frac{m}{m-1}} - 1}{\left(\left(1 + \frac{F''(x_1^*)}{|F'(x_1^*)|^2 + F''(x_1^*)} \right)^{\frac{1}{m-1}} - 1 \right)^m} = \frac{(1 + b(x_1^*))^{\frac{m}{m-1}} - 1}{\left((1 + b(x_1^*))^{\frac{1}{m-1}} - 1 \right)^m} \\ &\leq C_m \left(\frac{1}{b(x_1^*)} \right)^{m-1} = C_m \left(\frac{|F'(x_1^*)|^2 + F''(x_1^*)}{F''(x_1^*)} \right)^{m-1} = C_m \left(1 + \frac{|F'(x_1^*)|^2}{F''(x_1^*)} \right)^{m-1}, \end{aligned}$$

where in the last inequality we used (1) the fact that $b(x_1^*) = \frac{F''(x_1^*)}{|F'(x_1^*)|^2 + F''(x_1^*)} < 1$ provided $x_1^* \leq r$, which we may assume since otherwise we are done, and (2) the inequality

$$\frac{(1+b)^{\frac{m}{m-1}} - 1}{\left((1+b)^{\frac{1}{m-1}} - 1\right)^m} \leq \frac{1}{2} m(2m-1)(m-1)^{2m} b^{1-m}, \quad 0 \leq b < 1,$$

which follows easily from upper and lower estimates on the binomial series. Combining this with (7.28) we thus obtain the following upper bound

$$\mathcal{F}(x_1) \leq t_m |F'(r_0)| \frac{1}{|F'(x_1^*)|} e^{C_m \left(1 + \frac{|F'(x_1^*)|^2}{F''(x_1^*)}\right)^{m-1}} = t_m |F'(r_0)| \varphi(x_1^*),$$

with φ as in (7.21). Using the monotonicity of φF we therefore obtain

$$\mathcal{I}_{0, \frac{1}{|F'(x_1)|}}(x) \lesssim \mathcal{F}(x_1) \leq t_m |F'(r_0)| \varphi(r_0) = t_m |F'(r_0)| \varphi(r_0),$$

which is the estimate required in (7.25).

For the second big region \mathcal{R}_2 we have

$$\frac{1}{h_{y_1-x_1}} \approx \frac{|F'(x_1+r)|}{f(x_1+r)},$$

and the integral to be estimated becomes

$$I_{\mathcal{R}_2} \equiv \frac{1}{\omega(r_0)} \int_{x_1 + \frac{1}{|F'(x_1)|}}^{r_0} \left(\frac{f(r_0)|F'(y_1)|}{f(y_1)|F'(r_0)|^2 \omega(r_0)} \right)^{\psi\left(\frac{f(r_0)|F'(y_1)|}{f(y_1)|F'(r_0)|^2 \omega(r_0)}\right)} dy_1.$$

We still have the condition (7.27) for this integral, i.e.

$$(7.29) \quad A(y_1) |F'(y_1)| = \frac{f(r_0)}{f(y_1)|F'(r_0)|^2 \omega(r_0)} |F'(y_1)| \geq E.$$

Again, we would like to estimate the above integral by $C_m \varphi(r_0) |F'(r_0)|$.

For this we introduce the change of variables

$$\begin{aligned} y_1 &\rightarrow v := \frac{f(r_0)|F'(y_1)|}{f(y_1)\omega(r_0)|F'(r_0)|^2} \\ dv &= -\frac{|F'(y_1)|^2 - F''(y_1)}{f(y_1)} \frac{f(r_0)}{\omega(r_0)|F'(r_0)|^2} dy_1 \end{aligned}$$

We can also assume that

$$|F'(y_1)|^2 - F''(y_1) \approx |F'(y_1)|^2,$$

for small enough y_1 which gives

$$dy_1 \approx -\frac{1}{|F'(y_1)|} \frac{dv}{v},$$

and we rewrite the integral as

$$\int_{v_0}^{v_1} \frac{1}{\omega(r_0)|F'(y_1)|} v^{\psi(v)-1} dv,$$

where we denoted by v_0 and v_1 values of v corresponding to $y_1 = r_0$ and $y_1 = x_1 + \frac{1}{|F'(x_1)|}$ respectively.

Now we make a few observations. First, we already assumed that $v \geq E$ on the whole range of integration. Since

$$v_0 = v(r_0) = \frac{1}{\omega(r_0)|F'(r_0)|} = t_m \geq E,$$

we may assume that the range of integration starts at $v = E$. Next, without loss of generality we will assume that x_1 is such that $v(x_1) > E$. Then we have

$$\int_E^{v_1} \frac{v^{\psi(v)-1}}{\omega(r_0)|F'(y_1)|} dv = \int_E^{v_1} \frac{v^{2\psi(v)}}{\omega(r_0)|F'(y_1)|} \frac{dv}{v^{1+\psi(v)}}.$$

Recalling the definition of v we write

$$\frac{v^{2\psi(v)}}{\omega(r_0)|F'(y_1)|} = \frac{1}{\omega(r_0)|F'(y_1)|} \left(\frac{f(r_0)|F'(y_1)|}{f(y_1)\omega(r_0)|F'(r_0)|^2} \right)^{2\psi \frac{f(r_0)|F'(y_1)|}{f(y_1)\omega(r_0)|F'(r_0)|^2}}.$$

Next, denote

$$\begin{aligned} \mathcal{G}(y_1) &\equiv \frac{1}{\omega(r_0)|F'(y_1)|} \left(\frac{f(r_0)|F'(y_1)|}{f(y_1)\omega(r_0)|F'(r_0)|^2} \right)^{2\psi \frac{f(r_0)|F'(y_1)|}{f(y_1)\omega(r_0)|F'(r_0)|^2}} \\ &= t_m |F'(r_0)| \frac{1}{|F'(x_1)|} \left(c(r_0) \frac{|F'(x_1)|}{f(x_1)} \right)^{2\psi \left(c(r_0) \frac{|F'(x_1)|}{f(x_1)} \right)}, \end{aligned}$$

where

$$c(r_0) = \frac{t_m f(r_0)}{|F'(r_0)|},$$

and look for the maximum of $\mathcal{G}(y_1)$ on $(0, r_0]$. Note that the only difference between functions $\mathcal{G}(t)$ and $\mathcal{F}(t)$ defined in (7.28) is an additional coefficient of 2 in the exponential.

We claim that a bound for \mathcal{G} can be obtained in a similar way and yields

$$\mathcal{G}(y_1) \leq C_m |F'(r_0)| \varphi(r_0),$$

where $\varphi(r_0)$ satisfies (7.21) with a constant C_m slightly bigger than in the case of \mathcal{F} . Indeed, rewriting $\mathcal{G}(y_1)$ in a form similar to (7.28) we have

$$\mathcal{G}(y_1) = t_m |F'(r_0)| \frac{1}{|F'(y_1)|} \exp \left(2 \left(1 + \left(\ln \left[c(r_0) \frac{|F'(y_1)|}{f(y_1)} \right] \right)^{\frac{1}{m}} \right)^m - 2 \ln \left[c(r_0) \frac{|F'(y_1)|}{f(y_1)} \right] \right)$$

Again, we differentiate and equate the derivative to zero to obtain the following implicit expression for y_1^* maximizing $\mathcal{G}(y_1)$:

$$\frac{F''(y_1^*)}{|F'(y_1^*)|^2} = 2 \left(\left(1 + \left(\ln \left[c(r_0) \frac{|F'(y_1^*)|}{f(y_1^*)} \right] \right)^{-\frac{1}{m}} \right)^{m-1} - 1 \right) \left(1 + \frac{F''(y_1^*)}{|F'(y_1^*)|^2} \right).$$

A calculation similar to the one for the function \mathcal{F} gives

$$\begin{aligned} \left(1 + \left(\ln \left[c(r_0) \frac{|F'(y_1^*)|}{f(y_1^*)}\right]\right)^{\frac{1}{m}}\right)^m - \ln \left[c(r_0) \frac{|F'(y_1^*)|}{f(y_1^*)}\right] &= \frac{\left(1 + \frac{1}{2} \frac{F''(y_1^*)}{|F'(y_1^*)|^2 + F''(y_1^*)}\right)^{\frac{m}{m-1}} - 1}{\left(\left(1 + \frac{1}{2} \frac{F''(y_1^*)}{|F'(y_1^*)|^2 + F''(y_1^*)}\right)^{\frac{1}{m-1}} - 1\right)^m} \\ &\leq C_m \left(\frac{|F'(y_1^*)|^2 + F''(y_1^*)}{F''(y_1^*)}\right)^{m-1} = C_m \left(1 + \frac{|F'(y_1^*)|^2}{F''(y_1^*)}\right)^{m-1}, \end{aligned}$$

with C_m larger than before. From this and the monotonicity condition we conclude

$$\mathcal{G}(y_1) \leq C_m |F'(r_0)| \varphi(r_0).$$

The bound for the integral therefore becomes

$$I_{\mathcal{R}_2} \leq C_m |F'(r_0)| \varphi(r_0) \int_E^{v_1} \frac{dv}{v^{1+\psi(v)}},$$

where

$$\int_E^{v_1} \frac{dv}{v^{1+\psi(v)}} \approx \sum_{n=0}^N (e^n E)^{-\psi(e^n E)} \quad \text{with } N \approx \ln \frac{v_1}{E}.$$

Using the definition of ψ we have

$$\psi(e^n E) = \left[\ln(e^n E)^{-\frac{1}{m}} + 1 \right]^m - 1 \approx \frac{1}{n^{\frac{1}{m}}},$$

and thus

$$\sum_{n=0}^N (e^n E)^{-\psi(e^n E)} \approx \sum_{n=0}^{\infty} e^{-n^{\frac{m-1}{m}}} < C.$$

This concludes the estimate for the region \mathcal{R}_2

$$I_{\mathcal{R}_2} \leq C_m |F'(r_0)| \varphi(r_0),$$

which is (7.25). ■

Now we turn to the proof of Corollary 93.

PROOF OF COROLLARY 93. We must first check that the monotonicity property (7.21) holds for the indicated geometries $F_{k,\sigma}$, where

$$\begin{aligned} f(r) &= f_{k,\sigma}(r) \equiv \exp \left\{ - \left(\ln \frac{1}{r} \right) \left(\ln^{(k)} \frac{1}{r} \right)^{\sigma} \right\}; \\ F(r) &= F_{k,\sigma}(r) \equiv \left(\ln \frac{1}{r} \right) \left(\ln^{(k)} \frac{1}{r} \right)^{\sigma}. \end{aligned}$$

Consider first the case $k = 1$. Then $F(r) = F_{1,\sigma}(r) = \left(\ln \frac{1}{r} \right)^{1+\sigma}$ satisfies

$$F'(r) = -(1+\sigma) \frac{\left(\ln \frac{1}{r} \right)^{\sigma}}{r} \quad \text{and} \quad F''(r) = -(1+\sigma) \left\{ -\frac{\left(\ln \frac{1}{r} \right)^{\sigma}}{r^2} - \sigma \frac{\left(\ln \frac{1}{r} \right)^{\sigma-1}}{r^2} \right\},$$

which shows that

$$\begin{aligned}\varphi(r) &= \frac{1}{1+\sigma} \exp \left\{ -\ln \frac{1}{r} - \sigma \ln \ln \frac{1}{r} + C_m \left(\frac{(1+\sigma)^2 \frac{(\ln \frac{1}{r})^{2\sigma}}{r^2}}{(1+\sigma) \left\{ \frac{(\ln \frac{1}{r})^\sigma}{r^2} + \sigma \frac{(\ln \frac{1}{r})^{\sigma-1}}{r^2} \right\}} + 1 \right)^{m-1} \right\} \\ &= \frac{1}{1+\sigma} \exp \left\{ -\ln \frac{1}{r} - \sigma \ln \ln \frac{1}{r} + C_m (1+\sigma)^{m-1} \left(\frac{(\ln \frac{1}{r})^\sigma}{\left\{ 1 + \sigma \frac{1}{\ln \frac{1}{r}} \right\}} + \frac{1}{1+\sigma} \right)^{m-1} \right\},\end{aligned}$$

is increasing in r provided both $\sigma(m-1) < 1$ and $0 \leq r \leq \alpha_{m,\sigma}$, where $\alpha_{m,\sigma}$ is a positive constant depending only on m and σ . Hence we have the upper bound

$$\varphi(r) \leq \exp \left\{ -\ln \frac{1}{r} + C_m \left(\ln \frac{1}{r} \right)^{\sigma(m-1)} \right\} = r^{1-C_m \frac{1}{(\ln \frac{1}{r})^{1-\sigma(m-1)}}}, \quad 0 \leq r \leq \beta_{m,\sigma},$$

where $\beta_{m,\sigma} > 0$ is chosen even smaller than $\alpha_{m,\sigma}$ if necessary.

Thus in the case $\Phi = \Phi_m$ with $m > 2$ and $F = F_\sigma$ with $0 < \sigma < \frac{1}{m-1}$, we see that the norm $\varphi(r_0)$ of the Sobolev embedding satisfies

$$\varphi(r_0) \leq r_0^{1-C_m \frac{1}{(\ln \frac{1}{r_0})^{1-\sigma(m-1)}}}, \quad \text{for } 0 < r_0 \leq \beta_{m,\sigma},$$

and hence that

$$\frac{\varphi(r_0)}{r_0} \leq \left(\frac{1}{r_0} \right)^{\frac{C_m}{(\ln \frac{1}{r_0})^{1-\sigma(m-1)}}} \quad \text{for } 0 < r_0 \leq \beta_{m,\sigma}.$$

Now consider the case $k \geq 2$. Our first task is to show that $F_{k,\sigma}$ satisfies the structure conditions in Definition 14. Only condition (5) is not obvious, so we now turn to that. We have $F(r) = F_{k,\sigma}(r) = \left(\ln \frac{1}{r} \right) \left(\ln^{(k)} \frac{1}{r} \right)^\sigma$ satisfies

$$\begin{aligned}F'(r) &= -\frac{\left(\ln^{(k)} \frac{1}{r} \right)^\sigma}{r} - \left(\ln \frac{1}{r} \right) \frac{\sigma \left(\ln^{(k)} \frac{1}{r} \right)^{\sigma-1}}{\left(\ln^{(k-1)} \frac{1}{r} \right) \left(\ln^{(k-2)} \frac{1}{r} \right) \dots \left(\ln \frac{1}{r} \right) r} \\ &= -\frac{\left(\ln^{(k)} \frac{1}{r} \right)^\sigma}{r} - \frac{\sigma \left(\ln^{(k)} \frac{1}{r} \right)^{\sigma-1}}{\left(\ln^{(k-1)} \frac{1}{r} \right) \left(\ln^{(k-2)} \frac{1}{r} \right) \dots \left(\ln^{(2)} \frac{1}{r} \right) r} \\ &= -\frac{\left(\ln^{(k)} \frac{1}{r} \right)^\sigma}{r} \left\{ 1 + \frac{\sigma}{\left(\ln^{(k)} \frac{1}{r} \right) \left(\ln^{(k-1)} \frac{1}{r} \right) \left(\ln^{(k-2)} \frac{1}{r} \right) \dots \left(\ln^{(2)} \frac{1}{r} \right)} \right\} \\ &= -\frac{F(r)}{r \ln \frac{1}{r}} \left\{ 1 + \frac{\sigma}{\left(\ln^{(k)} \frac{1}{r} \right) \left(\ln^{(k-1)} \frac{1}{r} \right) \dots \left(\ln^{(2)} \frac{1}{r} \right)} \right\} \equiv -\frac{F(r) \Lambda_k(r)}{r \ln \frac{1}{r}},\end{aligned}$$

and

$$\begin{aligned} F''(r) &= -\frac{F'(r) \Lambda_k(r)}{r \ln \frac{1}{r}} - \frac{F(r) \Lambda'_k(r)}{r \ln \frac{1}{r}} - F(r) \Lambda_k(r) \frac{d}{dr} \left(\frac{1}{r \ln \frac{1}{r}} \right) \\ &= -\frac{F'(r) \Lambda_k(r)}{r \ln \frac{1}{r}} - \frac{F(r) \Lambda'_k(r)}{r \ln \frac{1}{r}} + \frac{F(r) \Lambda_k(r)}{r^2 \ln \frac{1}{r}} \left(1 - \frac{1}{\ln \frac{1}{r}} \right), \end{aligned}$$

where

$$\begin{aligned} \Lambda'_k(r) &= \frac{d}{dr} \left(\frac{\sigma}{\left(\ln^{(k)} \frac{1}{r} \right) \left(\ln^{(k-1)} \frac{1}{r} \right) \dots \left(\ln^{(2)} \frac{1}{r} \right)} \right) \\ &= -\sigma \sum_{j=2}^k \frac{\left(\ln^{(j)} \frac{1}{r} \right)}{\left(\ln^{(k)} \frac{1}{r} \right) \dots \left(\ln^{(2)} \frac{1}{r} \right) \left(\ln^{(j-1)} \frac{1}{r} \right) \dots \left(\ln \frac{1}{r} \right) r} \frac{1}{\left(\ln^{(j-1)} \frac{1}{r} \right) \dots \left(\ln \frac{1}{r} \right) r} \\ &= -\sigma \frac{1}{\left(\ln^{(k)} \frac{1}{r} \right) \dots \left(\ln^{(2)} \frac{1}{r} \right) r} \sum_{j=2}^k \frac{\ln^{(j)} \frac{1}{r}}{\left(\ln^{(j-1)} \frac{1}{r} \right) \dots \left(\ln \frac{1}{r} \right)} \\ &= -\sigma \frac{1}{\left(\ln^{(k)} \frac{1}{r} \right) \dots \left(\ln^{(2)} \frac{1}{r} \right) r} \left(\frac{\ln^{(2)} \frac{1}{r}}{\ln \frac{1}{r}} + \sum_{j=3}^k \frac{\ln^{(j)} \frac{1}{r}}{\left(\ln^{(j-1)} \frac{1}{r} \right) \dots \left(\ln \frac{1}{r} \right)} \right) \\ &= -\sigma \frac{1}{\left(\ln^{(k)} \frac{1}{r} \right) \dots \left(\ln^{(2)} \frac{1}{r} \right) \left(\ln \frac{1}{r} \right) r} \left(\ln^{(2)} \frac{1}{r} + \sum_{j=3}^k \frac{\ln^{(j)} \frac{1}{r}}{\left(\ln^{(j-1)} \frac{1}{r} \right) \dots \left(\ln^{(2)} \frac{1}{r} \right)} \right). \end{aligned}$$

Now

$$\ln^{(2)} \frac{1}{r} + \sum_{j=3}^k \frac{\ln^{(j)} \frac{1}{r}}{\left(\ln^{(j-1)} \frac{1}{r} \right) \dots \left(\ln^{(2)} \frac{1}{r} \right)} \approx \ln^{(2)} \frac{1}{r},$$

and so

$$-\Lambda'_k(r) \approx \begin{cases} \frac{\sigma}{\left(\ln \frac{1}{r} \right) r} & \text{for } k = 2 \\ \frac{\sigma}{\left(\ln^{(k)} \frac{1}{r} \right) \dots \left(\ln^{(3)} \frac{1}{r} \right) \left(\ln \frac{1}{r} \right) r} & \text{for } k \geq 3 \end{cases}.$$

We also have $\Lambda_k(r) \approx 1$, which then gives

$$-F'(r) \approx \frac{F(r)}{r \ln \frac{1}{r}},$$

and

$$F''(r) \approx \frac{F(r)}{r^2 \left(\ln \frac{1}{r} \right)^2} + \frac{\sigma F(r)}{\left(\ln^{(k)} \frac{1}{r} \right) \dots \left(\ln^{(3)} \frac{1}{r} \right) \left(\ln \frac{1}{r} \right)^2 r^2} + \frac{F(r)}{r^2 \ln \frac{1}{r}} \approx \frac{F(r)}{r^2 \ln \frac{1}{r}}.$$

From these two estimates we immediately obtain structure condition (5) of Definition 14.

We also have

$$\frac{|F'(r)|^2}{F''(r)} \approx \frac{F(r)^2}{\left(r \ln \frac{1}{r} \right)^2} \frac{r^2 \ln \frac{1}{r}}{F(r)} = \frac{F(r)}{\ln \frac{1}{r}} = \left(\ln^{(k)} \frac{1}{r} \right)^\sigma, \quad 0 \leq r \leq \beta_{m,\sigma},$$

and then from the definition of $\varphi(r) \equiv \frac{1}{|F'(r)|} e^{C_m \left(\frac{|F'(r)|^2}{F''(r)} + 1 \right)^{m-1}}$ in (7.21), we obtain

$$\begin{aligned} \varphi(r) &= \frac{1}{|F'(r)|} e^{C_m \left(\frac{|F'(r)|^2}{F''(r)} + 1 \right)^{m-1}} \approx r \frac{e^{C_m \left(\ln^{(k)} \frac{1}{r} \right)^{\sigma(m-1)}}}{\left(\ln^{(k)} \frac{1}{r} \right)^{\sigma}} \\ &\lesssim r e^{C_m \left(\ln^{(k)} \frac{1}{r} \right)^{\sigma(m-1)}} \approx r^{1-C_m \frac{\left(\ln^{(k)} \frac{1}{r} \right)^{\sigma(m-1)}}{\ln \frac{1}{r}}}, \quad 0 \leq r \leq \beta_{m,\sigma}. \end{aligned}$$

This completes the proof of the monotonicity property (7.21) and the estimates for $\varphi(r)$ for each of the two cases in Corollary 93.

Finally, we must show that the standard (Φ, φ) -Sobolev inequality (7.14) with Φ as in (7.19), $m > 1$, fails if $k = 1$ and $\sigma > \frac{1}{m-1}$, and for this it is convenient to use the identity $|\nabla_A v| = \left| \frac{\partial v}{\partial r} \right|$ for radial functions v . To see this identity, we recall that in Region 1 of the plane as defined in Section 3 above we have

$$\begin{aligned} \frac{\partial(x, y)}{\partial(r, \lambda)} &= \begin{bmatrix} \frac{\sqrt{\lambda^2 - f(x)^2}}{\frac{f(x)^2}{\lambda}} & \frac{\sqrt{\lambda^2 - f(x)^2}}{\frac{f(x)^2 - \lambda^2}{\lambda}} m_3(x) \end{bmatrix}; \\ \text{where } m_3(x) &= \int_0^x \frac{f(u)^2}{\left(\lambda^2 - f(u)^2 \right)^{\frac{3}{2}}} du. \end{aligned}$$

Then $\det \left(\frac{\partial(x, y)}{\partial(r, \lambda)} \right) = -\sqrt{\lambda^2 - f(x)^2} m_3(x)$ and

$$\begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \lambda}{\partial x} & \frac{\partial \lambda}{\partial y} \end{bmatrix} = \frac{\partial(r, \lambda)}{\partial(x, y)} = \begin{bmatrix} \frac{\sqrt{\lambda^2 - f(x)^2}}{\frac{f(x)^2}{\lambda}} & \frac{1}{\lambda} \\ \frac{f(x)^2}{\lambda \sqrt{\lambda^2 - f(x)^2} m_3(x)} & -\frac{1}{\lambda m_3(x)} \end{bmatrix}.$$

Thus if $v = v(r)$ is radial, then

$$\begin{aligned} \nabla_A v &= \left(\frac{\partial v}{\partial x}, f(x) \frac{\partial v}{\partial y} \right) = \left(\frac{\sqrt{\lambda^2 - f(x)^2}}{\lambda} \frac{\partial v}{\partial r}, f(x) \frac{1}{\lambda} \frac{\partial v}{\partial r} \right) \\ &= \left(\sqrt{1 - \left(\frac{f(x)}{\lambda} \right)^2}, \frac{f(x)}{\lambda} \right) \frac{\partial v}{\partial r}, \end{aligned}$$

and so in Region 1, $|\nabla_A v| = \left| \frac{\partial v}{\partial r} \right|$ for radial v - even though $\nabla_A v$ is not in general radial. In Region 2 we have

$$\frac{\partial(x, y)}{\partial(r, \lambda)} = \begin{bmatrix} -\frac{\sqrt{\lambda^2 - f(x)^2}}{\frac{f(x)^2}{\lambda}} & \frac{2\sqrt{\lambda^2 - f(x)^2}}{\lambda} R'(\lambda) + \frac{\sqrt{\lambda^2 - f(x)^2}}{\frac{f(x)^2 - \lambda^2}{\lambda}} m_3(x) \\ 2Y'(\lambda) - \frac{2f(x)^2}{\lambda} R'(\lambda) + \frac{\lambda^2 - f(x)^2}{\lambda} m_3(x) \end{bmatrix}.$$

To simplify this expression, recall

$$R(\lambda) = \int_0^{f^{-1}(\lambda)} \frac{\lambda}{\sqrt{\lambda^2 - f(u)^2}} du \text{ and } Y(\lambda) = \int_0^{f^{-1}(\lambda)} \frac{f(u)^2}{\sqrt{\lambda^2 - f(u)^2}} du,$$

so that we have

$$\begin{aligned}\lambda R(\lambda) - Y(\lambda) &= \int_0^{f^{-1}(\lambda)} \sqrt{\lambda^2 - f(u)^2} du; \\ \text{implies } Y'(\lambda) &= \lambda R'(\lambda).\end{aligned}$$

Thus we can write

$$J \equiv \frac{\partial(x, y)}{\partial(r, \lambda)} = \begin{bmatrix} -\frac{\sqrt{\lambda^2 - f(x)^2}}{\lambda} & \frac{\sqrt{\lambda^2 - f(x)^2}}{\lambda} (2R'(\lambda) + m_3(x)) \\ \frac{f(x)^2}{\lambda} & \frac{\lambda^2 - f(x)^2}{\lambda} (2R'(\lambda) + m_3(x)) \end{bmatrix}$$

and we have

$$\det J = -\sqrt{\lambda^2 - f(x)^2} (2R'(\lambda) + m_3(x))$$

and

$$\frac{\partial(r, \lambda)}{\partial(x, y)} = \frac{1}{\det J} \begin{bmatrix} \frac{\lambda^2 - f(x)^2}{\lambda} (2R'(\lambda) + m_3(x)) & -\frac{\sqrt{\lambda^2 - f(x)^2}}{\lambda} (2R'(\lambda) + m_3(x)) \\ -\frac{f(x)^2}{\lambda} & -\frac{\sqrt{\lambda^2 - f(x)^2}}{\lambda} \end{bmatrix} = \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ * & * \end{bmatrix}$$

We now calculate

$$|\nabla_A r|^2 = \left(\frac{\partial r}{\partial x} \right)^2 + f(x)^2 \left(\frac{\partial r}{\partial y} \right)^2 = \frac{(\lambda^2 - f(x)^2) (2R'(\lambda) + m_3(x))^2}{(\det J)^2} = 1$$

which again implies $|\nabla_A v| = \left| \frac{\partial v}{\partial r} \right|$ for a radial function $v = v(r)$.

Now we take $f(r) = f_{1,\sigma}(r) = r^{(\ln \frac{1}{r})^\sigma}$ and with $\eta(r) \equiv \begin{cases} 1 & \text{if } 0 \leq r \leq \frac{r_0}{2} \\ 2\left(1 - \frac{r}{r_0}\right) & \text{if } \frac{r_0}{2} \leq r \leq r_0 \end{cases}$, we

define the radial function

$$w(x, y) = w(r) = e^{(\ln \frac{1}{r})^{\sigma+1}} = \frac{\eta(r)}{f(r)}, \quad 0 < r < r_0.$$

From $|\nabla_A r| = 1$, we obtain the equality $|\nabla_A w(x, y)| = |\nabla_A r| |w'(r)| = |w'(r)|$, and combining this with $|\nabla_A \eta(r)| \leq \frac{2}{r_0} \mathbf{1}_{[\frac{r_0}{2}, r_0]}$ and the estimate (6.8), we have

$$\begin{aligned} \int \int_{B(0, r_0)} |\nabla_A w(x, y)| dx dy &\approx \int_0^{r_0} |w'(r)| \frac{f(r)}{|F'(r)|} dr + \frac{2}{r_0} \int_{\frac{r_0}{2}}^{r_0} \frac{1}{|F'(r)|} dr \\ &\approx \int_0^{r_0} \frac{f'(r)}{f(r)^2} \frac{f(r)^2}{f'(r)} dr + \frac{2}{r_0} \int_{\frac{r_0}{2}}^{r_0} C r dr \approx r_0. \end{aligned}$$

On the other hand, $\Phi_m(t) \geq t^{1 + \frac{m}{(\ln t)^{\frac{1}{m}}}}$ and $|F'(r)| = (\sigma + 1) (\ln \frac{1}{r})^\sigma \frac{1}{r}$, so we obtain

$$\begin{aligned} &\int \int_{B(0, r_0)} \Phi_m(w(x, y)) dx dy \\ &\gtrsim \int_0^{\frac{r_0}{2}} \Phi_m\left(\frac{1}{f(r)}\right) \frac{f(r)}{|F'(r)|} dr \geq \int_0^{\frac{r_0}{2}} \left(\frac{1}{f(r)}\right)^{1 + \frac{m}{F(r)^{\frac{1}{m}}}} \frac{f(r)}{|F'(r)|} dr \\ &\approx \int_0^{\frac{r_0}{2}} f(r)^{-\frac{m}{(\ln \frac{1}{r})^{\frac{\sigma}{m}}}} \frac{1}{(\ln \frac{1}{r})^\sigma \frac{1}{r}} dr = \int_0^{\frac{r_0}{2}} e^{m(\ln \frac{1}{r})^{(\sigma+1)(1-\frac{1}{m})}} \frac{r dr}{(\ln \frac{1}{r})^\sigma} = \infty \end{aligned}$$

if $(\sigma + 1)(1 - \frac{1}{m}) > 1$, i.e. $\sigma > \frac{1}{m-1}$. ■

5. Sobolev inequalities for supermultiplicative bumps when $t < \frac{1}{M}$

Here we prove a strong (Φ, φ) -Sobolev Orlicz bump inequality (7.15) that is needed to obtain continuity of weak solutions. However, the methods used in the previous section exploited a bump function $\Phi : [1, \infty] \rightarrow [1, \infty]$ defined for large values of the solution that satisfied the following three properties (although only the first two were actually used):

- (1) $\Phi(t)$ is closer to the identity function t than any power bump t^γ , $\gamma > 1$,
- (2) $\Phi(t)$ is convex,
- (3) $\Phi(t)$ is submultiplicative.

The function $\Phi_m(t) = e^{((\ln t)^{\frac{1}{m}} + 1)^m}$ continues to be submultiplicative even for small $t > 0$, but becomes concave for $0 < t < 1$. Now that we are interested in continuity of weak solutions, we need a bump function $\Phi : [0, 1] \rightarrow [0, 1]$ defined for small values of the solution. However, according to the next lemma, such a bump function on $[0, 1]$ cannot satisfy the three properties above unless $\Phi(t) = t$ is the identity.

LEMMA 94. *Suppose that $\Phi : [0, 1] \rightarrow [0, 1]$ is increasing and satisfies $\Phi(0) = 0$ and $\Phi(1) = 1$, and also satisfies the three properties listed above. Then $\Phi(t) = t$ is the identity function on $[0, 1]$.*

PROOF. Suppose that properties (2) and (3) hold and that Φ is not the identity function. We must show that property (1) fails. But from property (2) and $\Phi(0) = 0$ and $\Phi(1) = 1$, we have $\Phi(x) \leq x$ for $0 \leq x \leq 1$. Thus if Φ is not the identity function, then there is $x_0 < 1$ with $0 \leq \Phi(x_0) < x_0 < 1$. We may assume $\Phi(x_0) > 0$ (otherwise Φ vanishes identically on $[0, x_0]$) and we can then define $\gamma > 1$ by

$$\Phi(x_0) = x_0^\gamma.$$

Now by property (3) we have,

$$\Phi(x_0^n) \leq \Phi(x_0)^n = x_0^{n\gamma}$$

and since Φ increasing, we conclude that for $x_0^{n+1} \leq t \leq x_0^n$ we have

$$\Phi(t) \leq \Phi(x_0^n) \leq x_0^{n\gamma} = x_0^{-\gamma} x_0^{(n+1)\gamma} \leq x_0^{-\gamma} t^\gamma.$$

This shows that

$$\Phi(t) \leq x_0^{-\gamma} t^\gamma, \quad 0 \leq t \leq x_0,$$

which shows that property (1) fails. ■

Now we consider the case $\Phi : [0, 1] \rightarrow [0, b]$ with $b > 0$. Lemma 94 persists with $b \leq 1$. On the other hand, if $b > 1$ then the function $\tilde{\Phi}(t) \equiv \frac{1}{b}\Phi(t)$ satisfies the hypotheses of Lemma 94 except that $\tilde{\Phi}$ is now b -submultiplicative on $[0, 1]$. Now we run the proof of Lemma 94 with this assumption instead and obtain the following result.

LEMMA 95. *Suppose that $\tilde{\Phi} : [0, 1] \rightarrow [0, 1]$ is increasing and satisfies $\tilde{\Phi}(0) = 0$ and $\tilde{\Phi}(1) = 1$, and also satisfies the three properties listed above, except that property (3) is replaced with $\tilde{\Phi}(t)$ is b -submultiplicative. Then $\tilde{\Phi}(t) \approx t$.*

PROOF. Suppose that properties (1), (2) and (3) hold and that $\tilde{\Phi}$ is not the identity function. We must then show that $\tilde{\Phi}(t) \approx t$. But from property (2) and $\tilde{\Phi}(0) = 0$ and $\tilde{\Phi}(1) = 1$, we have $\tilde{\Phi}(x) \leq x$ for $0 \leq x \leq 1$. Thus if $\tilde{\Phi}$ is not the identity function, then there is $x_0 < 1$ with

$0 \leq \tilde{\Phi}(x_0) < x_0 < 1$. We may assume $\tilde{\Phi}(x_0) > 0$ (otherwise $\tilde{\Phi}$ vanishes identically on $[0, x_0]$ and property (1) fails, a contradiction) and we can then define $\sigma > 0$ by

$$(7.30) \quad \tilde{\Phi}(x_0) = x_0^{1+\sigma}.$$

Now by property (3) we have,

$$\begin{aligned} \tilde{\Phi}(x_0^n) &= \tilde{\Phi}(x_0 x_0^{n-1}) \leq b \tilde{\Phi}(x_0) \tilde{\Phi}(x_0^{n-1}) \leq \\ &\dots \leq b^n \tilde{\Phi}(x_0)^n = b^n x_0^{n(1+\sigma)} = \left(b x_0^{\frac{\sigma}{2}}\right)^n x_0^{n(1+\frac{\sigma}{2})}. \end{aligned}$$

At this point we wish to have in addition the inequality

$$(7.31) \quad b x_0^{\frac{\sigma}{2}} \leq 1,$$

so that using $\tilde{\Phi}$ increasing, we can conclude that for $x_0^{n+1} \leq t \leq x_0^n$ we have

$$\tilde{\Phi}(t) \leq \tilde{\Phi}(x_0^n) \leq x_0^{n(1+\frac{\sigma}{2})} = x_0^{-(1+\frac{\sigma}{2})} x_0^{(n+1)(1+\frac{\sigma}{2})} \leq C t^\gamma,$$

where $C = x_0^{-(1+\frac{\sigma}{2})}$ and $\gamma = 1 + \frac{\sigma}{2}$. This would then show that

$$\Phi(t) \leq C t^\gamma, \quad 0 \leq t \leq x_0,$$

which shows that property (1) fails, a contradiction.

But in order to obtain both (7.30) and (7.31), we need to solve the equation $\tilde{\Phi}(x_0) = x_0^{1+\sigma}$ with $0 < x_0 < 1$ and $0 < \sigma < 2$ that satisfy $b x_0^{\frac{\sigma}{2}} \leq 1$. Thus $0 < x_0 \leq \frac{1}{b^{\frac{2}{\sigma}}}$ and we need to know that

$$(7.32) \quad \tilde{\Phi}\left(\frac{1}{b^{\frac{2}{\sigma}}}\right) \leq \frac{1}{b^{\frac{2}{\sigma}(1+\sigma)}}, \quad \text{i.e.} \quad \frac{\tilde{\Phi}\left(\frac{1}{b^{\frac{2}{\sigma}}}\right)}{\frac{1}{b^{\frac{2}{\sigma}(1+\sigma)}}} \leq 1,$$

for some $0 < \sigma < 2$. Indeed, if this is true for some $0 < \sigma < 2$, then since $\lim_{x_0 \rightarrow 0} \frac{\tilde{\Phi}(x_0)}{x_0^{1+\sigma}} = \infty$ by property (1), we can use the intermediate value theorem to conclude that

$$\frac{\tilde{\Phi}(x_0)}{x_0^{1+\sigma}} = 1, \quad \text{i.e.} \quad \tilde{\Phi}(x_0) = x_0^{1+\sigma}, \quad \text{for some } 0 < x_0 \leq \frac{1}{b^{\frac{2}{\sigma}}}.$$

As σ ranges between 0 and 2, we see that $\frac{1}{b^{\frac{2}{\sigma}}}$ ranges between 0 to $\frac{1}{b}$, and if $t = \frac{1}{b^{\frac{2}{\sigma}}}$, then a calculation shows that

$$\frac{1}{b^{\frac{2}{\sigma}(1+\sigma)}} = \frac{t}{b^2},$$

and so to establish (7.32), we must find some t with $0 < t \leq \frac{1}{b}$ such that

$$\tilde{\Phi}(t) \leq \frac{t}{b^2}, \quad \text{i.e.} \quad \frac{\tilde{\Phi}(t)}{t} \leq \frac{1}{b^2}.$$

But now if $\lim_{t \rightarrow 0} \frac{\tilde{\Phi}(t)}{t} = 0$, we see that we can indeed find $0 < t \leq \frac{1}{b}$ such that $\frac{\tilde{\Phi}(t)}{t} \leq \frac{1}{b^2}$, which by the above argument shows that $\tilde{\Phi}$ is the identity function - contradicting the assumption just made that $\lim_{t \rightarrow 0} \frac{\tilde{\Phi}(t)}{t} = 0$. So we conclude that we must have $\lim_{t \rightarrow 0} \frac{\tilde{\Phi}(t)}{t} \neq 0$, and the convexity of $\tilde{\Phi}$ now shows that $\tilde{\Phi}(t) \approx t$. ■

Finally we record one more trivial extension of this result.

LEMMA 96. *Suppose that $\Phi : [0, a] \rightarrow [0, a]$ is increasing and satisfies $\Phi(0) = 0$ and $\Phi(a) = a$, and also satisfies the three properties listed above, except that property (3) is replaced with $\Phi(t)$ is b -submultiplicative. If in addition $\lim_{t \rightarrow 0} \frac{\Phi(t)}{t} = 0$, then $\Phi(t) \approx t$ on $[0, a]$.*

In view of these considerations, we define here for $m > 1$ and $0 < t < \frac{1}{M}$,

$$(7.33) \quad \begin{aligned} \Psi_m(t) &= Ae^{-\left(\ln \frac{1}{t}\right)^{\frac{1}{m}+1}} = At^{\left(\ln \frac{1}{t}\right)^{-\frac{1}{m}+1}}; \\ A &= e^{\left(\ln M\right)^{\frac{1}{m}+1} - \ln M} > 1, \end{aligned}$$

and we extend Ψ to be linear on $[\frac{1}{M}, \infty)$ with slope $\Psi'(\frac{1}{M})$. Note that

$$\begin{aligned} \Psi(t) &= At^{1+\psi(t)}; \\ \psi(t) &\equiv \left(\left(\ln \frac{1}{t} \right)^{-\frac{1}{m}} + 1 \right)^m - 1 \approx \frac{m}{\ln\left(\frac{1}{t}\right)^{\frac{1}{m}}}, \quad 0 < t < \frac{1}{M}, \end{aligned}$$

and that

$$(7.34) \quad \Psi^{(-1)}(s) \leq s^{1-\psi(s)}.$$

Indeed, we compute

$$\begin{aligned} s &= \Psi(t) = Ae^{-\left(\ln \frac{1}{t}\right)^{\frac{1}{m}+1}}; \\ \left(\ln \frac{A}{s} \right)^{\frac{1}{m}} &= \left(\ln \frac{1}{t} \right)^{\frac{1}{m}} + 1; \\ \Psi^{(-1)}(s) &= t = e^{-\left(\ln \frac{A}{s}\right)^{\frac{1}{m}-1}} = \left(\frac{s}{A} \right)^{\left(1 - \left(\ln \frac{A}{s}\right)^{-\frac{1}{m}}\right)^m}, \end{aligned}$$

and since $\Psi^{(-1)}$ is increasing and $A > 1$, we have

$$\Psi^{(-1)}(s) \leq \Psi^{(-1)}(As) = s^{\left(1 - \left(\ln \frac{1}{s}\right)^{-\frac{1}{m}}\right)^m}.$$

Then we note that

$$\left(1 - \left(\ln \frac{1}{s} \right)^{-\frac{1}{m}} \right)^m \geq 2 - \left(1 + \left(\ln \frac{1}{s} \right)^{-\frac{1}{m}} \right)^m$$

for all $m \geq 1$ since

$$(1+x)^m + (1-x)^m \geq 2, \quad m \geq 1.$$

It thus follows that

$$\Psi^{(-1)}(s) \leq s^{\left(1 - \left(\ln \frac{1}{s}\right)^{-\frac{1}{m}}\right)^m} \leq s^{2 - \left(1 + \left(\ln \frac{1}{s}\right)^{-\frac{1}{m}}\right)^m} = s^{1-\psi(s)}.$$

Note also that the function ψ here satisfies $\psi(t) \approx \frac{m}{\ln\left(\frac{1}{t}\right)^{\frac{1}{m}}}$ for small $t > 0$, while the corresponding function ψ in the previous section satisfied $\psi(t) \approx \frac{m}{\ln(t)^{\frac{1}{m}}}$ for large $t > 0$.

Finally, we point out that from Lemma 102 below, the (Ψ_m, φ) -Sobolev Orlicz bump inequality (see (7.15)) cannot hold with $\varphi(r) = O(r)$.

5.1. The inhomogeneous Sobolev Orlicz bump inequality.

PROPOSITION 97. *Let $0 < r_0 < 1$ and $B = B(0, r_0)$. Let $w \in W_0^{1,2}(B)$ and let $\Psi = \Psi_m$ be as in (7.33) and suppose the geometry F satisfies the following monotonicity property*

$$(7.35) \quad r^{-\varepsilon} \varphi(r) \text{ is an increasing function of } r \text{ for some } \varepsilon > 0,$$

where

$$(7.36) \quad \varphi(r) \equiv r e^{C_m \left(\frac{|F'(r)|^2}{F''(r)} + 1 \right)^{m-1}}.$$

Then the inhomogeneous Orlicz-Sobolev inequality (7.14) holds with Ψ in place of Φ , i.e.

$$\Psi^{-1} \left(\int_B \Psi(w) \right) \leq C \varphi(r(B)) \int_B \|\nabla_A w\| d\mu, \quad w \in Lip_0(B),$$

and moreover, the strong (Ψ, φ) -Sobolev Orlicz bump inequality (7.15) holds.

Note that the only differences between the superradius $\varphi(r)$ in Proposition 97 here, and the superradius $\varphi(r) = \frac{1}{|F'(r)|} e^{C_m \left(\frac{|F'(r)|^2}{F''(r)} + 1 \right)^{m-1}}$ in Proposition 92 earlier, is that the constants C_m may be different and that the ratio of the terms r and $\frac{1}{|F'(r)|}$ in front of the exponentials satisfies $0 < \varepsilon \leq r|F'(r)| < \infty$ by property (4) of Definition 14, and may be unbounded.

COROLLARY 98. *The strong (Ψ, φ) -Sobolev inequality with $\Psi = \Psi_m$ as in (7.33), $m > 1$, and geometry $F = F_{k,\sigma}$ where $F_{k,\sigma}(r) \equiv \left(\ln \frac{1}{r} \right) \left(\ln^{(k)} \frac{1}{r} \right)^\sigma$ holds if **(either)** $k \geq 2$ and $\sigma > 0$ and $\varphi(r_0)$ is given by*

$$\varphi(r_0) = r_0^{1 - C_m \frac{\left(\ln^{(k)} \frac{1}{r_0} \right)^\sigma (m-1)}{\ln \frac{1}{r_0}}}, \quad \text{for } 0 < r_0 \leq \beta_{m,\sigma},$$

for positive constants C_m and $\beta_{m,\sigma}$ depending only on m and σ ;

(or) $k = 1$ and $\sigma < \frac{1}{m-1}$ and $\varphi(r_0)$ is given by

$$\varphi(r_0) = r_0^{1 - C_m \frac{1}{\left(\ln \frac{1}{r_0} \right)^{1-\sigma(m-1)}}}, \quad \text{for } 0 < r_0 \leq \beta_{m,\sigma},$$

for positive constants C_m and $\beta_{m,\sigma}$ depending only on m and σ .

Now we turn to the proof of Proposition 97.

PROOF. Using the subrepresentation inequality we see that it is enough to show

$$(7.37) \quad \Psi^{-1} \left(\sup_{y \in B} \int_B \Psi(K(x, y)|B|^\alpha) d\mu(x) \right) \leq C \alpha \varphi(r),$$

for all $\alpha > 0$. Indeed, if (7.37) holds, then with $g = \|\nabla_A(w)\|$ and $\alpha = \|g\|_{L^1}$ in (7.37), we have

$$\begin{aligned} \int_B \Psi(w) d\mu(x) &\lesssim \int_B \Psi \left(\int_B K(x, y) |B| \|g\|_{L^1(\mu)} \frac{g(y) d\mu(y)}{\|g\|_{L^1(\mu)}} \right) d\mu(x) \\ &\leq \int_B \int_B \Psi \left(K(x, y) |B| \|g\|_{L^1(\mu)} \right) \frac{g(y) d\mu(y)}{\|g\|_{L^1(\mu)}} d\mu(x) \\ &\leq \int_B \left\{ \sup_{y \in B} \int_B \Psi \left(K(x, y) |B| \|g\|_{L^1(\mu)} \right) d\mu(x) \right\} \frac{g(y) d\mu(y)}{\|g\|_{L^1(\mu)}} \\ &\leq \Psi(\varphi(r) \|g\|_{L^1(\mu)}) \int_B \frac{g(y) d\mu(y)}{\|g\|_{L^1(\mu)}} = \Psi(C(\varphi(r) \|g\|_{L^1(\mu)})), \end{aligned}$$

and so

$$\Psi^{-1} \left(\int_B \Psi(w) d\mu(x) \right) \lesssim C\varphi(r) \int_B |\nabla_A(w)| d\mu.$$

Again, we will show (7.37) with x and y interchanged. From now on we take $B \equiv B(0, r_0)$. First, recall

$$|B(0, r_0)| \approx \frac{f(r_0)}{|F'(r_0)|^2},$$

and

$$K(x, y) \approx \frac{1}{h_{y_1 - x_1}} \approx \begin{cases} \frac{1}{rf(x_1)}, & r = y_1 - x_1 < \frac{1}{|F'(x_1)|} \\ \frac{|F'(x_1 + r)|}{f(x_1 + r)}, & r = y_1 - x_1 \geq \frac{1}{|F'(x_1)|} \end{cases}.$$

Next, recall that

$$\Psi(t) = At^{1+\psi(t)}$$

for small t , where $\psi(t) = \left[1 + (\ln \frac{1}{t})^{-\frac{1}{m}}\right]^m - 1 \approx \frac{m}{\ln(1/t)^{1/m}}$. We then have

$$\int_{B(0, r_0)} \Psi \left(K_{B(0, r_0)}(x, y) |B(0, r_0)| \alpha \right) \frac{dy}{|B(0, r_0)|} \approx \int_0^{r_0} \alpha \left(\frac{|B(0, r_0)| \alpha}{h_{y_1 - x_1}} \right)^{\psi \left(\frac{|B(0, r_0)| \alpha}{h_{y_1 - x_1}} \right)} dy_1,$$

and in order to obtain (7.37), we must dominate this last integral by $\Psi(C\alpha\varphi(r))$.

Now divide the interval of integration into three regions:

- (1): the big region \mathcal{L} where the integrand $K_{B(0, r_0)}(x, y) |B(0, r_0)| \alpha \geq 1/M$,
- (2): the region \mathcal{R}_1 disjoint from \mathcal{L} where $r = x_1 - y_1 < 1/|F'(x_1)|$ and
- (3): the region \mathcal{R}_2 disjoint from \mathcal{L} where $r = x_1 - y_1 \geq 1/|F'(x_1)|$.

We turn first to the region \mathcal{R}_1 where we have $h_{y_1 - x_1} \approx rf(x_1)$. We claim the following, which is the desired estimate for the integral over region \mathcal{R}_1 :

$$(7.38) \quad \int_{\mathcal{R}_1} \alpha \left(\frac{|B(0, r_0)| \alpha}{h_{y_1 - x_1}} \right)^{\psi \left(\frac{|B(0, r_0)| \alpha}{h_{y_1 - x_1}} \right)} dy_1 \lesssim M\alpha\varphi(r_0).$$

The integral that we want to estimate is thus

$$\int_{AM}^{\frac{1}{|F'(x_1)|}} \alpha \left(\frac{f(r_0)\alpha}{rf(x_1)|F'(r_0)|^2} \right)^{\psi \left(\frac{f(r_0)\alpha}{rf(x_1)|F'(r_0)|^2} \right)} dr = \int_{AM}^{\frac{1}{|F'(x_1)|}} \alpha \left(\frac{A}{r} \right)^{\psi \left(\frac{A}{r} \right)} dr$$

where $A = A(x_1) \equiv \frac{f(r_0)\alpha}{f(x_1)|F'(r_0)|^2}$.

Making a change of variables

$$R = \frac{A}{r} = \frac{A(x_1)}{r},$$

we obtain

$$\int_{AM}^{\frac{1}{|F'(x_1)|}} \alpha \left(\frac{A}{r} \right)^{\psi\left(\frac{A}{r}\right)} dr = \alpha A \int_{A|F'(x_1)|}^{1/M} R^{\psi(R)-2} dR$$

Integrating by parts gives

$$\begin{aligned} \int_{A|F'(x_1)|}^{1/M} R^{\psi(R)-2} dR &= \int_{A|F'(x_1)|}^{1/M} R^{\psi(R)+1} \left(-\frac{1}{2R^2} \right)' dR \\ &= -\frac{R^{\psi(R)+1}}{2R^2} \Big|_{A|F'(x_1)|}^{1/M} + \int_{A|F'(x_1)|}^{1/M} \left(R^{\psi(R)+1} \right)' \frac{1}{2R^2} dR \\ &\leq \frac{(A|F'(x_1)|)^{\psi(A|F'(x_1)|)}}{2A|F'(x_1)|} + \int_{A|F'(x_1)|}^{1/M} \frac{1}{2} R^{\psi(R)-2} \left(1 + C \frac{m-1}{(\ln \frac{1}{R})^{\frac{1}{m}}} \right) dR \\ &\leq \frac{(A|F'(x_1)|)^{\psi(A|F'(x_1)|)}}{2A|F'(x_1)|} + \frac{1 + C \frac{m-1}{(\ln M)^{\frac{1}{m}}}}{2} \int_{A|F'(x_1)|}^{1/M} R^{\psi(R)-2} dR, \end{aligned}$$

where we used

$$|\psi'(R)| \leq C \frac{1}{R} \frac{1}{(\ln \frac{1}{R})^{\frac{m+1}{m}}}.$$

Taking M large enough depending on m we can assure

$$\frac{1 + C \frac{m-1}{(\ln M)^{\frac{1}{m}}}}{2} \leq \frac{3}{4},$$

which gives

$$\int_{A|F'(x_1)|}^{1/M} R^{\psi(R)-2} dR \lesssim \frac{(A|F'(x_1)|)^{\psi(A|F'(x_1)|)}}{A|F'(x_1)|},$$

and therefore

$$\begin{aligned} (7.39) \quad \alpha A \int_{A|F'(x_1)|}^{1/M} R^{\psi(R)-2} dR &\lesssim \frac{\alpha}{|F'(x_1)|} (A|F'(x_1)|)^{\psi(A|F'(x_1)|)} \\ &= \Psi(A(x_1) |F'(x_1)|) \frac{f(x_1) |F'(r_0)|^2}{f(r_0) |F'(x_1)|^2}. \end{aligned}$$

Now if the factor $\frac{f(x_1) |F'(r_0)|^2}{f(r_0) |F'(x_1)|^2}$ is greater than $\frac{1}{M}$, then it is easy to obtain the bound we want. Indeed,

$$\begin{aligned} A(x_1) |F'(x_1)| &= \frac{f(r_0) \alpha}{f(x_1) |F'(r_0)|^2} |F'(x_1)| \\ &\leq \frac{f(r_0) |F'(x_1)|^2}{f(x_1) |F'(r_0)|^2} \frac{\alpha}{|F'(x_1)|} \leq \frac{M \alpha}{|F'(x_1)|}, \end{aligned}$$

gives

$$\begin{aligned} \alpha A \int_{A|F'(x_1)|}^{1/M} R^{\psi(R)-2} dR &\lesssim \Psi(A(x_1)|F'(x_1)|) \frac{f(x_1)|F'(r_0)|^2}{f(r_0)|F'(x_1)|^2} \\ &\leq \Psi\left(\frac{M\alpha}{|F'(x_1)|}\right) \leq \Psi\left(\frac{M\alpha}{|F'(r_0)|}\right), \end{aligned}$$

which proves (7.38) in this case, since $\frac{1}{|F'(r_0)|} \leq \varphi(r_0)$.

On the other hand, if $\frac{f(x_1)|F'(r_0)|^2}{f(r_0)|F'(x_1)|^2} < \frac{1}{M}$, then applying $\Psi^{(-1)}$ on both sides of (7.39), and using the submultiplicativity of $\Psi^{(-1)}(t)$ on the interval $(0, \frac{1}{M})$, we get

$$\begin{aligned} \Psi^{(-1)}\left(\alpha A \int_{A|F'(x_1)|}^{1/M} R^{\psi(R)-2} dR\right) &\lesssim A|F'(x_1)|\Psi^{(-1)}\left(\frac{f(x_1)|F'(r_0)|^2}{f(r_0)|F'(x_1)|^2}\right) \\ &\leq \frac{f(r_0)\alpha|F'(x_1)|}{f(x_1)|F'(r_0)|^2} \left(\frac{f(x_1)|F'(r_0)|^2}{f(r_0)|F'(x_1)|^2}\right)^{1-\psi\left(\frac{f(x_1)|F'(r_0)|^2}{f(r_0)|F'(x_1)|^2}\right)} \\ &= \alpha \frac{\left(\frac{f(x_1)|F'(r_0)|^2}{f(r_0)|F'(x_1)|^2}\right)^{-\psi\left(\frac{f(x_1)|F'(r_0)|^2}{f(r_0)|F'(x_1)|^2}\right)}}{|F'(x_1)|}, \end{aligned}$$

where the middle inequality follows from the estimate $\Psi^{(-1)}(s) \leq s^{1-\psi(s)}$ in (7.34).

To conclude the proof of (7.38) in this case, we need to show that

$$(7.40) \quad \sup_{x_1 \in (0, r_0)} \frac{\left(\frac{f(x_1)|F'(r_0)|^2}{f(r_0)|F'(x_1)|^2}\right)^{-\psi\left(\frac{f(x_1)|F'(r_0)|^2}{f(r_0)|F'(x_1)|^2}\right)}}{|F'(x_1)|} \leq \varphi(r_0).$$

As in the previous section we define an auxiliary function

$$\mathcal{F}(x_1) \equiv \frac{1}{|F'(x_1)|} \left(\frac{f(x_1)|F'(r_0)|^2}{f(r_0)|F'(x_1)|^2}\right)^{-\psi\left(\frac{f(x_1)|F'(r_0)|^2}{f(r_0)|F'(x_1)|^2}\right)} = \frac{1}{|F'(x_1)|} \left(\frac{1}{c(r_0)} \frac{f(x_1)}{|F'(x_1)|^2}\right)^{-\psi\left(\frac{1}{c(r_0)} \frac{f(x_1)}{|F'(x_1)|^2}\right)}$$

where

$$c(r_0) = \frac{f(r_0)}{|F'(r_0)|^2}.$$

Again, we would like to find the maximum of $\mathcal{F}(x_1)$ on $(0, r_0)$. First, rewrite the expression for $\mathcal{F}(x_1)$ using the definition of $\psi(t)$

$$\psi(t) = \left(\left(\ln \frac{1}{t}\right)^{-\frac{1}{m}} + 1\right)^m - 1,$$

to obtain

$$\mathcal{F}(x_1) = \frac{1}{|F'(x_1)|} \exp\left(\left(1 + \left(\ln \left[c(r_0) \frac{|F'(x_1)|^2}{f(x_1)}\right]\right)^{\frac{1}{m}}\right)^m - \ln \left[c(r_0) \frac{|F'(x_1)|^2}{f(x_1)}\right]\right).$$

Note that this expression is very similar to (7.28) except for $|F'(x_1)|$ being squared in the argument of the exponential.

We thus proceed in the same way we did right after definition (7.28) and skip most details, recording only the main steps. Differentiating $\mathcal{F}(x_1)$ with respect to x_1 and then setting the derivative equal to zero, we obtain

$$\frac{F''(x_1^*)}{|F'(x_1^*)|^2} = \left(\left(1 + \left(\ln \left[c(r_0) \frac{|F'(x_1^*)|}{f(x_1^*)} \right] \right)^{-\frac{1}{m}} \right)^{m-1} - 1 \right) \left(1 + 2 \frac{F''(x_1^*)}{|F'(x_1^*)|^2} \right).$$

as an implicit expression for x_1^* which maximizes $\mathcal{F}(x_1)$. Denoting

$$B \equiv \ln \left[c(r_0) \frac{|F'(x_1^*)|}{f(x_1^*)} \right],$$

we obtain from above

$$\begin{aligned} \left(1 + B^{\frac{1}{m}} \right)^m - B &= \frac{\left(1 + \frac{F''(x_1^*)}{|F'(x_1^*)|^2 + 2F''(x_1^*)} \right)^{\frac{m}{m-1}} - 1}{\left(\left(1 + \frac{F''(x_1^*)}{|F'(x_1^*)|^2 + 2F''(x_1^*)} \right)^{\frac{1}{m-1}} - 1 \right)^m} \\ &\leq C_m \left(\frac{|F'(x_1^*)|^2 + 2F''(x_1^*)}{F''(x_1^*)} \right)^{m-1} = C_m \left(2 + \frac{|F'(x_1^*)|^2}{F''(x_1^*)} \right)^{m-1}. \end{aligned}$$

Now note that we can obtain a weaker bound from above in order to directly use the monotonicity property (7.35), namely,

$$C_m \left(2 + \frac{|F'(x_1^*)|^2}{F''(x_1^*)} \right)^{m-1} = C_m 2^{m-1} \left(1 + \frac{1}{2} \frac{|F'(x_1^*)|^2}{F''(x_1^*)} \right)^{m-1} \leq \tilde{C}_m \left(1 + \frac{|F'(x_1^*)|^2}{F''(x_1^*)} \right)^{m-1}$$

where $\tilde{C}_m > C_m$. Finally, we obtain as before

$$\begin{aligned} \mathcal{F}(x_1) &\leq \mathcal{F}(x_1^*) = \frac{1}{|F'(x_1^*)|} \exp \left(\left(1 + B^{\frac{1}{m}} \right)^m - B \right) \\ &\leq \frac{(x_1^*)^{\varepsilon-1}}{|F'(x_1^*)|} (x_1^*)^{1-\varepsilon} e^{C_m \left(1 + \frac{|F'(x_1^*)|^2}{F''(x_1^*)} \right)^{m-1}} \leq \varphi(r_0), \end{aligned}$$

where the last inequality follows from the monotonicity property (7.35) and assumption (5) on the geometry F from Chapter 6. This concludes the proof of (7.40) and thus the estimate for region \mathcal{R}_1 .

For the region \mathcal{R}_2 , the integral to be estimated is

$$I_{\mathcal{R}_2} \equiv \int_{x_1+M}^{r_0} \alpha \left(\frac{f(r_0)|F'(y_1)|\alpha}{f(y_1)|F'(r_0)|^2} \right)^{\psi \left(\frac{f(r_0)|F'(y_1)|\alpha}{f(y_1)|F'(r_0)|^2} \right)} dy_1$$

where

$$M = \max \left\{ \frac{1}{|F'(x_1)|}, A \right\}$$

Again, we would like to estimate the above integral by $\Psi(C\alpha\varphi(r_0))$. First, we rewrite the integral above as follows

$$\begin{aligned} I_{\mathcal{R}_2} &= \int_{x_1+M}^{r_0} \Psi \left(\frac{f(r_0)|F'(y_1)|\alpha}{f(y_1)|F'(r_0)|^2} \right) \cdot \frac{f(y_1)|F'(r_0)|^2}{f(r_0)|F'(y_1)|} dy_1 \\ &= \int_{x_1+M}^{r_0} \Psi \left(\frac{f(r_0)|F'(y_1)|\alpha}{f(y_1)|F'(r_0)|^2} \right) \cdot \frac{f(y_1)|F'(r_0)|^2\varphi(r_0)}{f(r_0)|F'(y_1)|} \frac{dy_1}{\varphi(r_0)}, \end{aligned}$$

and recall that we would like to show

$$(7.41) \quad I_{\mathcal{R}_2} \leq \Psi(C\alpha\varphi(r_0)).$$

Now consider two cases. If $\frac{f(y_1)|F'(r_0)|^2\varphi(r_0)}{f(r_0)|F'(y_1)|} \geq \frac{1}{M}$, then we obtain the easy estimate

$$\begin{aligned} \Psi \left(\frac{f(r_0)|F'(y_1)|\alpha}{f(y_1)|F'(r_0)|^2} \right) &= \Psi \left(\frac{f(r_0)|F'(y_1)|}{f(y_1)|F'(r_0)|^2\varphi(r_0)} \alpha\varphi(r_0) \right) \\ &\leq \Psi(M\alpha\varphi(r_0)). \end{aligned}$$

Therefore,

$$\begin{aligned} I_{\mathcal{R}_2} &\leq \Psi(M\alpha\varphi(r_0)) \int_{x_1+M}^{r_0} \frac{f(y_1)|F'(r_0)|^2}{f(r_0)|F'(y_1)|} dy_1 \\ &= \Psi(M\alpha\varphi(r_0)) \int_{x_1+M}^{r_0} \frac{f'(y_1)|F'(r_0)|^2}{f(r_0)|F'(y_1)|^2} dy_1 \leq \Psi(M\alpha\varphi(r_0)), \end{aligned}$$

where we used $f'(y_1) = f(y_1)|F'(y_1)|$ and the fact that $|F'(y_1)|$ is a decreasing function of y_1 . On the other hand, if

$$(7.42) \quad \frac{f(y_1)|F'(r_0)|^2\varphi(r_0)}{f(r_0)|F'(y_1)|} < \frac{1}{M},$$

we can use supermultiplicativity of Ψ in the form

$$\Psi(X)Y = \Psi(X)\Psi(Y) \cdot \frac{Y}{\Psi(Y)} \leq \Psi(XY)Y^{-\psi(Y)}$$

with

$$X = \frac{f(r_0)|F'(y_1)|\alpha}{f(y_1)|F'(r_0)|^2}, \quad Y = \frac{f(y_1)|F'(r_0)|^2\varphi(r_0)}{f(r_0)|F'(y_1)|}$$

to obtain

$$\begin{aligned} I_{\mathcal{R}_2} &\leq \Psi(\alpha\varphi(r_0)) \int_{x_1+M}^{r_0} \left(\frac{f(y_1)|F'(r_0)|^2\varphi(r_0)}{f(r_0)|F'(y_1)|} \right)^{-\psi\left(\frac{f(y_1)|F'(r_0)|^2\varphi(r_0)}{f(r_0)|F'(y_1)|}\right)} \frac{dy_1}{\varphi(r_0)} \\ (7.43) \quad &= \Psi(M\alpha\varphi(r_0)) \int_{x_1+M}^{r_0} y_1^{1-\varepsilon} \left(\frac{f(y_1)|F'(r_0)|^2\varphi(r_0)}{f(r_0)|F'(y_1)|} \right)^{-\psi\left(\frac{f(y_1)|F'(r_0)|^2\varphi(r_0)}{f(r_0)|F'(y_1)|}\right)} \frac{dy_1}{y_1^{1-\varepsilon}\varphi(r_0)}. \end{aligned}$$

As before, we maximize the function

$$\begin{aligned} (7.44) \quad \mathcal{G}(y_1) &\equiv y_1^{1-\varepsilon} \left(\frac{1}{c(r_0)} \frac{f(y_1)}{|F'(y_1)|} \right)^{-\psi\left(\frac{1}{c(r_0)} \frac{f(y_1)}{|F'(y_1)|}\right)} \\ &= y_1^{1-\varepsilon} \exp \left(\left(1 + \left(\ln \left[c(r_0) \frac{|F'(y_1)|}{f(y_1)} \right] \right)^{\frac{1}{m}} \right)^m - \ln \left[c(r_0) \frac{|F'(y_1)|}{f(y_1)} \right] \right) \end{aligned}$$

for $y_1 \in (0, r_0)$, where $c(r_0) = (|F'(r_0)|^2 \varphi(r_0)/f(r_0))$. The value of y_1^* that maximizes $\mathcal{G}(y_1)$ satisfies

$$(1 - \varepsilon)(y_1^*)^{-\varepsilon} = \left(\left(1 + \left(\ln \left[c(r_0) \frac{|F'(y_1^*)|}{f(y_1^*)} \right] \right)^{-\frac{1}{m}} \right)^{m-1} - 1 \right) (y_1^*)^{1-\varepsilon} \left(|F'(y_1^*)| + \frac{F''(y_1^*)}{|F'(y_1^*)|} \right).$$

This gives for $B \equiv \ln \left[c(r_0) \frac{|F'(y_1^*)|}{f(y_1^*)} \right]$ the estimate

$$\begin{aligned} \left(1 + B^{\frac{1}{m}} \right)^m - B &= \frac{\left(1 + \frac{1-\varepsilon}{y_1 |F'(y_1)| + \frac{y_1 F''(y_1)}{|F'(y_1)|}} \right)^{\frac{m}{m-1}} - 1}{\left(\left(1 + \frac{1-\varepsilon}{y_1 |F'(y_1)| + \frac{y_1 F''(y_1)}{|F'(y_1)|}} \right)^{\frac{1}{m-1}} - 1 \right)^m} \\ &\leq \frac{C_m}{(1-\varepsilon)^{\frac{m}{m-1}}} \left(y_1 |F'(y_1)| + \frac{y_1 F''(y_1)}{|F'(y_1)|} \right)^{m-1} \\ &\leq \frac{\tilde{C}_m}{(1-\varepsilon)^{\frac{m}{m-1}}} \left(\frac{|F'(y_1)|^2}{F''(y_1)} + 1 \right)^{m-1}, \end{aligned}$$

where in the last inequality we used $|F'(r)/F''(r)| \approx r$. We therefore obtain

$$\mathcal{G}(y_1) \leq \mathcal{G}(y_1^*) = (y_1^*)^{1-\varepsilon} \exp \left(\left(1 + B^{\frac{1}{m}} \right)^m - B \right) \leq (y_1^*)^{1-\varepsilon} e^{\frac{C_m}{(1-\varepsilon)^{\frac{m}{m-1}}} \left(\frac{|F'(y_1)|^2}{F''(y_1)} + 1 \right)^{m-1}} \leq \frac{\varphi(r_0)}{r_0^\varepsilon}$$

where in the last inequality we used the monotonicity assumption (7.35) and the definition

$$\varphi(r_0) = r_0 e^{\frac{C_m}{(1-\varepsilon)^{\frac{m}{m-1}}} \left(\frac{|F'(r_0)|^2}{F''(r_0)} + 1 \right)^{m-1}}.$$

Thus we conclude from (7.43)

$$I_{\mathcal{R}_2} \leq \Psi(\alpha \varphi(r_0)) \int_{x_1+M}^{r_0} \frac{\varphi(r_0)}{r_0^\varepsilon} \frac{dy_1}{y_1^{1-\varepsilon} \varphi(r_0)} \leq \Psi(\alpha \varphi(r_0)) C_\varepsilon \leq \Psi(\tilde{C}_\varepsilon \alpha \varphi(r_0))$$

which is (7.41).

To finish the proof we need to estimate the integral over the big region \mathcal{L} . Recall that $\Psi(t)$ is affine for $t > 1/M$, more precisely, we have

$$\begin{aligned} \Psi(t) &= at - b \\ a &= \left(1 + (\ln M)^{-\frac{1}{m}} \right)^{m-1} > 1 \\ b &= \frac{1}{M} \left(\left(1 + (\ln M)^{-\frac{1}{m}} \right)^{m-1} - 1 \right) < a. \end{aligned}$$

Therefore, the required estimate (7.37) becomes

$$a\alpha \int_{y_1: (x_1, y_1) \in \mathcal{L}} dy_1 - b \frac{|\mathcal{L}|}{|B(0, r_0)|} \leq \Psi(C\varphi(r_0)\alpha).$$

Now note that if $C\varphi(r_0)\alpha > 1/M$ then $\Psi(C\varphi(r_0)\alpha) = aC\varphi(r_0)\alpha - b$ and the estimate (7.37) follows easily from the ‘straight across’ estimate (7.8). We therefore assume $C\varphi(r_0)\alpha \leq 1/M$ and note that

it is enough to show

$$(7.45) \quad \alpha \int_{y_1: (x_1, y_1) \in \mathcal{L}} dy_1 \leq \Psi(C\varphi(r_0)\alpha).$$

We now divide the region \mathcal{L} into two pieces, namely \mathcal{L}_1 where $K(x, y) \approx 1/rf(x_1)$, and \mathcal{L}_2 where $K(x, y) \approx \frac{|F'(y_1)|}{f(y_1)}$. In \mathcal{L}_1 the condition $K_{B(0, r_0)}(x, y) |B(0, r_0)| \alpha \geq 1/M$ becomes

$$r \leq \frac{M|B(0, r_0)|\alpha}{f(x_1)}.$$

Denote by r_α the value of r that gives equality in the above, i.e.

$$(7.46) \quad r_\alpha = \frac{M|B(0, r_0)|\alpha}{f(x_1)}.$$

Then the integral on the left hand side of (7.45) restricted to \mathcal{L}_1 can be written as

$$\alpha \int_{y_1: (x_1, y_1) \in \mathcal{L}_1} dy_1 = \alpha r_\alpha,$$

or using (7.46) to express α in terms of r_α , and the estimate $|B(0, r_0)| \approx \frac{f(r_0)}{|F'(r_0)|^2}$,

$$\alpha \int_{y_1: (x_1, y_1) \in \mathcal{L}_1} dy_1 \approx \frac{1}{M} r_\alpha^2 f(x_1) \frac{|F'(r_0)|^2}{f(r_0)}.$$

Inequality (7.45) for the region \mathcal{L}_1 therefore becomes

$$\frac{1}{M} r_\alpha^2 f(x_1) \frac{|F'(r_0)|^2}{f(r_0)} \leq \Psi\left(\frac{\varphi(r_0)}{M} r_\alpha f(x_1) \frac{|F'(r_0)|^2}{f(r_0)}\right)$$

or

$$(7.47) \quad \frac{r_\alpha}{C\varphi(r_0)} \left(\frac{\varphi(r_0)}{M} r_\alpha f(x_1) \frac{|F'(r_0)|^2}{f(r_0)}\right)^{-\psi\left(\frac{C\varphi(r_0)}{M} r_\alpha f(x_1) \frac{|F'(r_0)|^2}{f(r_0)}\right)} \leq 1.$$

It follows from the definition of ψ that the function $r(cr)^{-\psi(cr)}$ is an increasing function of r for all values of r such that $cr \leq 1/M$. For the left hand side of (7.47) we thus have an upper bound

$$\frac{r_\alpha^{max}}{C\varphi(r_0)} \left(\frac{1}{M}\right)^{-\psi\left(\frac{1}{M}\right)} \leq \frac{r_0}{C\varphi(r_0)} \left(\frac{1}{M}\right)^{-\psi\left(\frac{1}{M}\right)} \leq 1,$$

where the last inequality follows by choosing the constant C large enough depending on M . This shows (7.47) and therefore (7.45).

We now turn to region \mathcal{L}_2 where the condition $K_{B(0, r_0)}(x, y) |B(0, r_0)| \alpha \geq 1/M$ becomes

$$\frac{|F'(y_1)|}{f(y_1)} |B(0, r_0)| \alpha \geq \frac{1}{M}.$$

As above, denote by y_α the value of y_1 that gives equality, and obtain the following expression for α in terms of y_α

$$\alpha \approx \frac{1}{M} \frac{f(y_\alpha)}{|F'(y_\alpha)|} \frac{|F'(r_0)|^2}{f(r_0)}.$$

The estimate (7.45) for region \mathcal{L}_2 therefore becomes

$$\alpha y_\alpha = \frac{y_\alpha}{M} \frac{f(y_\alpha)}{|F'(y_\alpha)|} \frac{|F'(r_0)|^2}{f(r_0)} \leq \Psi \left(\frac{\varphi(r_0)}{M} \frac{f(y_\alpha)}{|F'(y_\alpha)|} \frac{|F'(r_0)|^2}{f(r_0)} \right)$$

or

$$(7.48) \quad \frac{y_\alpha}{C\varphi(r_0)} \left(\frac{\varphi(r_0)}{M} \frac{f(y_\alpha)}{|F'(y_\alpha)|} \frac{|F'(r_0)|^2}{f(r_0)} \right)^{-\psi \left(\frac{\varphi(r_0)}{M} \frac{f(y_\alpha)}{|F'(y_\alpha)|} \frac{|F'(r_0)|^2}{f(r_0)} \right)} \leq 1.$$

Now note that the expression on the left, when viewed as a function of y_α , has the form

$$\frac{1}{C\varphi(r_0)} \mathcal{G}(y_\alpha),$$

with $\mathcal{G}(y_\alpha)$ as in (7.44) with $\varepsilon = 0$ and a different constant $c(r_0)$. It has been shown above that the following bound holds for \mathcal{G} under the monotonicity assumption (7.35)

$$\mathcal{G}(y_\alpha) \leq r_0 e^{C_m \left(\frac{|F'(r_0)|^2}{F''(r_0)} + 1 \right)^{m-1}} = \varphi(r_0),$$

where we have put $\varepsilon = 0$. We thus obtain (7.48) and therefore (7.45). ■

Now we turn to the proof of Corollary 98.

PROOF OF COROLLARY 98. We must check that the monotonicity property (7.35) holds for the indicated geometries where

$$\begin{aligned} f(r) &= f_{k,\sigma}(r) \equiv \exp \left\{ - \left(\ln \frac{1}{r} \right) \left(\ln^{(k)} \frac{1}{r} \right)^\sigma \right\}; \\ F(r) &= F_{k,\sigma}(r) \equiv \left(\ln \frac{1}{r} \right) \left(\ln^{(k)} \frac{1}{r} \right)^\sigma. \end{aligned}$$

We have

$$r^{-\varepsilon} \varphi(r) = \exp \left\{ - (1-\varepsilon) \ln \frac{1}{r} + C_m \left(\frac{|F'(r)|^2}{F''(r)} + 1 \right)^{m-1} \right\},$$

which can be shown to be monotone by following the argument in the proof of Corollary 93. Thus in the case $\Psi = \Psi_m$ with $m > 2$ and $F = F_{1,\sigma}$ with $0 < \sigma < \frac{1}{m-1}$, we see that the superradius $\varphi(r_0)$ of the Sobolev embedding satisfies

$$\varphi(r_0) \leq r_0^{\frac{1-C_m}{\left(\ln \frac{1}{r_0} \right)^{1-\sigma(m-1)}}}, \quad \text{for } 0 < r_0 \leq \beta_{m,\sigma},$$

and hence that

$$\frac{\varphi(r_0)}{r_0} \leq \left(\frac{1}{r_0} \right)^{\frac{C_m}{\left(\ln \frac{1}{r_0} \right)^{1-\sigma(m-1)}}} \quad \text{for } 0 < r_0 \leq \beta_{m,\sigma}.$$

In the case $k \geq 2$ and $\sigma > 0$, we similarly obtain

$$\frac{\varphi(r_0)}{r_0} \leq \left(\frac{1}{r_0} \right)^{C_m \frac{\left(\ln^{(k)} \frac{1}{r_0} \right)^{\sigma(m-1)}}{\ln \frac{1}{r_0}}} \quad \text{for } 0 < r_0 \leq \beta_{m,\sigma},$$

and this completes the proof of Corollary 98. ■

CHAPTER 8

Geometric theorems in the plane

In this final chapter of the third part of the paper, we use our Sobolev inequalities for specific geometries to prove the geometric local boundedness and continuity theorems in the plane.

1. Local boundedness and maximum principle for weak subsolutions

Using the Inner Ball inequality in Theorem 43, together with Proposition 92, yields the following ‘geometric’ local boundedness result which proves Theorem 16 of the introduction.

THEOREM 99. *Suppose that u is a weak subsolution to (3.1) in a bounded open set $\Omega \subset \mathbb{R}^2$, i.e. $\mathcal{L}u = \nabla^{\text{tr}} A \nabla u = \phi$, where the matrix A satisfies (1.8), where ϕ is A -admissible, and where the degeneracy function $f = e^{-F}$ in (1.8) satisfies (7.21) in Proposition 92. In particular we can take $F = F_\sigma = (\ln \frac{1}{r})^{1+\sigma}$ with $0 < \sigma < 1$. Then u is locally bounded in Ω , i.e. for all compact subsets K of Ω ,*

$$\|u\|_{L^\infty(K)} \leq C'_K \left(1 + \|u\|_{L^2(\Omega)}\right),$$

and we have the maximum principle,

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u + \|\phi\|_{X(\Omega)}.$$

PROOF. Given $0 < \sigma < 1$, choose $m \in (2, 1 + \frac{1}{\sigma})$. Then $\sigma < \frac{1}{m-1}$, and so Proposition 92 shows that the inhomogeneous Φ -Sobolev bump inequality (7.14) holds with the near power bump Φ as in (7.19). Then since $m > 2$, Theorem 43 now shows that weak solutions to (3.1) are locally bounded.

In order to obtain the maximum principle, we need to establish the Φ_m -Sobolev inequality for Ω as given in (1.7). Of course, if $\Omega \subset B(x, r)$ for some $0 < r \leq R$, then this follows from Proposition 92. More generally, we need only construct a finite partition of unity $\{\eta_k\}_{k=1}^K$ for Ω consisting of Lipschitz functions η_k supported in metric balls $B(x_k, R)$. ■

2. Continuity of weak solutions for the geometries $F_{k,\sigma}$

We would like to obtain a sufficient condition on the geometry $F(x)$ that guarantees continuity of weak solutions provided all other assumptions hold. To this end we first note that we can take $\varphi(r)$ in the hypotheses of Theorem 5.61 to be the maximum of the superradii $\varphi(r)$ appearing in Corollaries 93 and 98. Then it remains only to verify the doubling increment growth condition (1.21) for this choice of $\varphi(r)$. For this we recall that

$$\begin{aligned} \delta(r) &\approx \frac{1}{|F'(r)|}, \\ \varphi(r) &= r e^{\tilde{C}_m(r|F'(r)|)^{m-1}}, \end{aligned}$$

and prove that the growth condition (1.21) holds for $\delta(r)$. But from $F(r) = F_{k,\sigma}(r) = -\ln \frac{1}{r} \left(\ln^{(k)} \frac{1}{r} \right)^\sigma$ we obtain

$$r|F'(r)| \approx \left(\ln^{(k)} \frac{1}{r} \right)^\sigma$$

and so

$$\ln \frac{\varphi(r)}{\delta(r)} \approx \ln^{(k+1)} \frac{1}{r} + \left(\ln^{(k)} \frac{1}{r} \right)^{\sigma(m-1)}.$$

Therefore we have

$$\left(\ln \frac{\varphi(r)}{\delta(r)} \right)^m = o \left(\ln^{(3)} \frac{1}{r} \right) \text{ as } r \rightarrow 0,$$

if

(either) $k \geq 4$ and $\sigma > 0$

(or) $k = 3$ and $\sigma < \frac{1}{m-1}$.

The above calculations complete the proof of Theorem 26.

Part 4

Sharpness of results

Here in this fourth part of the paper, we consider the extent to which our theorems above are sharp, first with respect to the Sobolev assumption, including the superradius, then with respect to the admissibility of the right hand side ϕ of the equation $\mathcal{L}u = \phi$, and finally with respect to the geometric assumptions made on the quadratic form associated with the operator \mathcal{L} in the left hand side of the equation. Our sharpness results for ϕ are quite tight, and correspond to the well known sharpness condition $\phi \in L^{\frac{n}{2}+\varepsilon}$ for elliptic operators \mathcal{L} in n -dimensional space. On the other hand, we have been unable to establish any sharpness with respect to the quadratic form of \mathcal{L} in the plane, and in \mathbb{R}^3 only with respect to local boundedness, even then leaving a large gap of an entire log between our sufficient and necessary conditions on the geometry.

CHAPTER 9

Examples in the plane

In this chapter, we first consider sharpness of Sobolev and superradius, and then we demonstrate a very weak degree of sharpness for our results. We now recall our inhomogeneous equation in which, under certain additional assumptions, we can obtain local boundedness and continuity of weak solutions. Consider the equation

$$\mathcal{L}u = \phi \text{ in } \Omega$$

where \mathcal{L} , A , and $f(x) = e^{-F(x)}$ are as above, and $\phi \in L^2(\Omega)$ for the moment. In order to determine the most reasonable conditions to impose on ϕ we will first look at the associated homogenous Dirichlet problem, where the most general condition presents itself.

1. Weak solutions to Dirichlet problems

Consider the homogeneous Dirichlet problem for $\mathcal{L} = \nabla^{\text{tr}} A \nabla$:

$$(9.1) \quad \begin{cases} \mathcal{L}u = \phi & \text{in } \Omega \\ u = 0 & \text{in } \partial\Omega \end{cases} .$$

We say $u \in W_A^{1,2}(\Omega)$ is a weak solution of $\mathcal{L}u = \phi$ in Ω if

$$-\int_{\Omega} \nabla u^{\text{tr}} A \nabla w = \int_{\Omega} \phi w, \quad w \in W_A^{1,2}(\Omega),$$

and we say u satisfies the boundary condition $u = 0$ in $\partial\Omega$ if $u \in \left(W_A^{1,2}\right)_0(\Omega)$. Thus altogether, $u \in \left(W_A^{1,2}\right)_0(\Omega)$ is a weak solution to (9.1) if and only if

$$-\int_{\Omega} \nabla u^{\text{tr}} A \nabla w = \int_{\Omega} \phi w, \quad w \in \left(W_A^{1,2}\right)_0(\Omega).$$

Now consider the bilinear form

$$B(v, w) \equiv \int_{\Omega} \nabla v^{\text{tr}} A \nabla w, \quad v, w \in \left(W_A^{1,2}\right)_0(\Omega),$$

on the Hilbert space $\left(W_A^{1,2}\right)_0(\Omega)$. This form is clearly bounded on $\left(W_A^{1,2}\right)_0(\Omega)$, and from the ‘straight across’ Sobolev inequality

$$\int_{\Omega} w^2 \leq C \int_{\Omega} |\nabla_A w|^2, \quad w \in \left(W_A^{1,2}\right)_0(\Omega),$$

we obtain that B is coercive:

$$B(w, w) = \int_{\Omega} |\nabla_A w|^2 \geq \frac{1}{2C} \int_{\Omega} |\nabla_A w|^2 + \frac{1}{2C} \int_{\Omega} w^2 = \frac{1}{2C} \|w\|_{\left(W_A^{1,2}\right)_0(\Omega)}^2 .$$

We now see that (9.1) has a weak solution $u \in \left(W_A^{1,2}\right)_0(\Omega)$ if and only if $\phi \in W_A^{1,2}(\Omega)^*$ where the dual is taken with respect to the pairing

$$[\phi, w] \equiv \int_{\Omega} \phi w.$$

Thus the elements ϕ of $W_A^{1,2}(\Omega)^*$ are distributions more general than L^2 functions. Indeed, if $\phi \in W_A^{1,2}(\Omega)^*$, then by definition, the linear functional

$$\Lambda_{\phi} w \equiv \int_{\Omega} \phi w, \quad w \in \left(W_A^{1,2}\right)_0(\Omega),$$

is bounded on $\left(W_A^{1,2}\right)_0(\Omega)$. So by the Lax-Milgram theorem applied to the coercive form B , there is $u \in \left(W_A^{1,2}\right)_0(\Omega)$ such that

$$-\int_{\Omega} \phi w = -\Lambda_{\phi} w = B(u, w) = \int_{\Omega} \nabla u^{\text{tr}} A \nabla w, \quad w \in \left(W_A^{1,2}\right)_0(\Omega),$$

which says that u is a weak solution to the homogeneous boundary value problem (9.1). Conversely, if $u \in \left(W_A^{1,2}\right)_0(\Omega)$ is a weak solution to (9.1), then

$$|\Lambda_{\phi} w| = \left| \int_{\Omega} \phi w \right| = \left| \int_{\Omega} \nabla u^{\text{tr}} A \nabla w \right| \leq \|u\|_{(W_A^{1,2})_0(\Omega)} \|w\|_{(W_A^{1,2})_0(\Omega)}$$

implies that Λ_{ϕ} is a bounded linear functional on $\left(W_A^{1,2}\right)_0(\Omega)$, i.e., that $\phi \in W_A^{1,2}(\Omega)^*$.

PROBLEM 100. *The homogeneous boundary value problem (9.1) is thus solvable in the weak sense when $\phi \in W_A^{1,2}(\Omega)^*$. Note that this space of linear functionals includes those induced by $L^2(\Omega)$ functions. Two natural questions now arise.*

- (1) *When are these weak solutions $u \in \left(W_A^{1,2}\right)_0(\Omega)$ locally bounded?*
- (2) *When are these weak solutions $u \in \left(W_A^{1,2}\right)_0(\Omega)$ continuous?*

In order to further investigate regularity of these weak solutions, we let Φ be an Orlicz bump (smaller than any power bump) for which the pair (Φ, F) satisfies an $L^1 \rightarrow L^{\Phi}$ Sobolev inequality. With $\tilde{\Phi}$ denoting the conjugate Young function to Φ , we now assume that

$$\phi \in L^{\tilde{\Phi}},$$

which is a weaker assumption than $\phi \in L^{\infty}$, but not as strong as $\phi \in L^q$ for all $q < \infty$. Recall that to make any progress on proving regularity of weak solutions *through the use of Caccioppoli's inequality*, we assumed above that the inhomogeneous term ϕ is A -admissible. Here we invoke the stronger condition that ϕ is strongly A -admissible as in Definition 32, i.e. there is a bump function Φ such that the Sobolev inequality holds for the control geometry associated with A and such that $\phi \in L^{\tilde{\Phi}}$ where $\tilde{\Phi}$ is the conjugate Young function to Φ . In the next section we will show that strong A -admissibility of ϕ is almost necessary for local boundedness of all weak solutions u to $\mathcal{L}u = \phi$.

Note that by Young's inequality applied to $K_x(y) = K(x, y)$ times ϕ , we have

$$\begin{aligned} \sup_{x \in B(0, r)} \left| \int_{B(0, r)} K_x \phi \right| &\leq \sup_{x \in B(0, r)} \|K_x\|_{L^\Phi(B(0, r))} \|\phi\|_{L^{\tilde{\Phi}}(B(0, r))} \\ &\approx \varphi(r) \|\phi\|_{L^{\tilde{\Phi}}(B(0, r))}. \end{aligned}$$

Thus the requirement that ϕ is strongly A -admissible implies that $T_{B(0, r)}\phi$ is bounded for all balls $B(z, r)$, but is in general much stronger than this. We remind the reader that the point of ϕ being strongly A -admissible is that we then have a Cacciopoli inequality for weak solutions to $\mathcal{L}u = \phi$. In this setting our regularity theorems become: If $F \approx F_\sigma$ with $0 < \sigma < 1$, and if ϕ is F_σ -admissible, then weak solutions u to $\mathcal{L}u = \phi$ are locally bounded. There is an analogous theorem for continuity of weak solutions u to $\mathcal{L}u = \phi$ when ϕ is strongly A -admissible.

2. Necessity of Sobolev inequalities for maximum principles and sharpness of the superradius

Here we show the necessity of the Orlicz Sobolev inequality for a strong form of the maximum principle, namely the global boundedness of $W_A^{1,2}$ -weak subsolutions to

$$(9.2) \quad \mathcal{L}u = \phi, \quad \phi \in \tilde{\Phi}(\Omega),$$

in a bounded open set Ω that vanish at the boundary $\partial\Omega$ in the sense that $u \in (W_A^{1,2})_0(\Omega)$.

PROPOSITION 101. *Suppose that for every $\phi \in \tilde{\Phi}(\Omega)$, and for every $W_A^{1,2}$ -weak subsolution u to (9.2) in Ω vanishing at the boundary in the sense that $u \in (W_A^{1,2})_0(\Omega)$, we have the global estimate*

$$(9.3) \quad \|u\|_{L^\infty(\Omega)} \leq C \|\phi\|_{\tilde{\Phi}(\Omega)}.$$

Then the following Orlicz Sobolev inequality holds for all functions $v \in Lip_0(\Omega)$:

$$(9.4) \quad \|v^2\|_{\tilde{\Phi}(\Omega)} \leq C' \|\nabla_A v\|_{L^2(\Omega)}.$$

The proof is almost exactly the same as the proof of Lemma 102 in [SaWh4], but we will record the main steps and point out the differences.

PROOF. Let $v \in Lip_0(\Omega)$. Then from (9.2) we have

$$\begin{aligned} \int_{\Omega} v^2 \phi &= \int_{\Omega} v^2 \nabla^{tr} A \nabla u = -2 \int_{\Omega} v \langle \nabla_A u, \nabla_A v \rangle \\ &\leq \left(\int_{\Omega} v^2 \|\nabla_A u\|^2 \right)^{\frac{1}{2}} \left(\int_{\Omega} \|\nabla_A v\|^2 \right)^{\frac{1}{2}} \end{aligned}$$

Following the proof of Lemma 102 in [SaWh4] we obtain

$$\int_{\Omega} v^2 \phi \leq C \left(\sup_{\Omega} |u| \right) \int_{\Omega} \|\nabla_A v\|^2 \leq C \|\phi\|_{\tilde{\Phi}(\Omega)} \int_{\Omega} \|\nabla_A v\|^2$$

where for the last inequality we used (9.3). We therefore have

$$\|v^2\|_{\tilde{\Phi}(\Omega)} = \sup_{\|\phi\|_{\tilde{\Phi}(\Omega)}=1} \left| \int_{\Omega} v^2 \phi \right| \leq C \int_{\Omega} \|\nabla_A v\|^2.$$

■

2.1. Sharpness of the superradius. Armed with Lemma 56 above, we can show that in the infinitely degenerate case for t small, the superradius $\varphi(r)$ in the (Ψ_m, φ) -Sobolev inequality with $m > 2$ must be asymptotically larger than r , i.e. that $\lim_{r \rightarrow 0} \frac{\varphi(r)}{r} = \infty$. Set $\widehat{\varphi}(r) = \frac{\varphi(r)}{r}$ for convenience.

LEMMA 102. *Fix $r_0 > 0$ and let $B_0 = B(0, r_0)$ and $B_1 = B(0, \frac{r_0}{2})$. If the single scale (Ψ_m, φ) -Sobolev inequality holds at scale r_0 for some $m > 2$, then*

$$\widehat{\varphi}(r_0) \geq B'_0(m, M, K, \gamma) \frac{|B_0|}{|B_1|}.$$

Since $\lim_{r_0 \rightarrow 0} \frac{|B_1|}{|B_0|} \approx \lim_{r_0 \rightarrow 0} \frac{f(\frac{r_0}{2})}{f(r_0)} = 0$ when f is infinitely degenerate, we immediately obtain as a corollary of the above lemma that we can never have $\varphi(r) = O(r)$ in the case of an infinitely degenerate geometry F . In fact for the geometries $F_{1,\sigma}$ we have the following corollary that shows our choice of superradius $\varphi(r) = r e^{C_m(\ln \frac{1}{r})^{\sigma(m-1)}}$ is essentially sharp up to $\varepsilon > 0$ arbitrarily small.

COROLLARY 103. *If the single scale (Ψ_m, φ) -Sobolev inequality holds at scale r_0 for some $m > 2$, and with the geometry $F_{1,\sigma}$, then we have*

$$\widehat{\varphi}(r_0) \gtrsim e^{c(\ln \frac{1}{r_0})^\sigma}.$$

PROOF. It suffices to observe that

$$\frac{|B_1|}{|B_0|} \approx \frac{f(\frac{r_0}{2})}{f(r_0)} \approx \frac{e^{-(\ln \frac{2}{r_0})^{1+\sigma}}}{e^{-(\ln \frac{1}{r_0})^{1+\sigma}}} = e^{-(\ln \frac{1}{r_0})^{1+\sigma} \left\{ \left(1 + \frac{\ln 2}{\ln \frac{1}{r_0}}\right)^{1+\sigma} - 1 \right\}} \approx e^{-(\ln \frac{1}{r_0})^\sigma}.$$

■

PROOF OF LEMMA 102. Let $\{\psi_j\}_{j=0}^\infty$ be a nonstandard sequence of Lipschitz cutoff functions at scale r_0 , and let $\{B_j\}_{j=0}^\infty$ be the corresponding sequence of balls. With $\Psi = \Psi_m$ we have

$$\begin{aligned} \gamma_{j+1} &\equiv \frac{|B_{j+1}|}{|B_0|} \Psi(1) \leq \int_{B_0} \Psi(\psi_j) \frac{dx}{|B_0|} \leq \Psi\left(\varphi(r_0) \int_{B_0} |\nabla_A \psi_j| \frac{dx}{|B_0|}\right) \\ &\leq \Psi\left(\widehat{\varphi}(r_0) j^2 \frac{|B_j|}{|B_0|}\right) = \Psi\left(\widehat{\varphi}(r_0) \frac{j^2}{\Psi(1)} \gamma_j\right), \end{aligned}$$

and iteration gives

$$\begin{aligned} \gamma_{j+1} &\leq \Psi\left(\widehat{\varphi}(r_0) \frac{j^2}{\Psi(1)} \gamma_j\right) \leq \Psi\left(\widehat{\varphi}(r_0) \frac{j^2}{\Psi(1)} \Psi\left(\widehat{\varphi}(r_0) \frac{(j-1)^2}{\Psi(1)} \gamma_{j-1}\right)\right) \\ \dots &\leq \Psi\left(\widehat{\varphi}(r_0) \frac{j^2}{\Psi(1)} \Psi\left(\widehat{\varphi}(r_0) \frac{(j-1)^2}{\Psi(1)} \dots \Psi\left(\widehat{\varphi}(r_0) \frac{|B_1|}{|B_0|}\right) \dots\right)\right). \end{aligned}$$

Then for $m > 2$, Lemma 56 gives the conclusion of Lemma 102, in view of the facts that $\inf_{j \geq 1} |B_j| > 0$ and $\lim_{j \rightarrow \infty} \Psi^{(j)}(C) = 0$ if $C = C(B_0, m, M, K, \gamma) < 1/M$ is the constant appearing in Lemma 56. ■

3. A discontinuous weak solution

Suppose that

- (1) $\psi(x)$ is smooth, even and strictly convex on \mathbb{R} , $\psi(0) = 0$, and so both $\psi(x)$ and $\psi'(x)$ are strictly increasing on $[0, \infty)$,
- (2) $\chi(s)$ is smooth and odd on \mathbb{R} , $\chi(s) = s$ for $s \in [-1, 1]$ and that $\chi(s) = 0$ for $s \in \mathbb{R} \setminus [-2, 2]$.

Then define

$$u(x, y) = \chi\left(\frac{y}{\psi(x)}\right), \quad x \neq 0.$$

Note that u fails to be continuous at the origin (this is where we use $\chi(s) = s$ for $s \in [-1, 1]$, but equality is not important, $\chi(s) \approx s$ will do). We compute with

$$s = s(x, y) = \frac{y}{\psi(x)},$$

that

$$\frac{\partial s}{\partial y} = \frac{1}{\psi(x)}, \quad \frac{\partial s}{\partial x} = -\frac{\psi'(x)y}{\psi(x)^2},$$

and

$$\frac{\partial^2 s}{\partial y^2} = 0, \quad \frac{\partial^2 s}{\partial x^2} = -\frac{\psi''(x)y}{\psi(x)^2} + \frac{2\psi(x)|\psi'(x)|^2 y}{(\psi(x)^2)^2},$$

so that

$$\mathcal{L}s = \frac{\partial^2 s}{\partial x^2} + |\psi'(x)|^2 \frac{\partial^2 s}{\partial y^2} = -\frac{\psi''(x)y}{\psi(x)^2} + \frac{2\psi(x)|\psi'(x)|^2 y}{(\psi(x)^2)^2}.$$

Now we compute the first derivatives of the composition $u = \chi \circ s$:

$$\begin{aligned} \frac{\partial}{\partial y} u(x, y) &= \chi'(s) \frac{\partial s}{\partial y} = \chi'(s) \frac{1}{\psi(x)}, \\ \frac{\partial}{\partial x} u(x, y) &= \chi'(s) \frac{\partial s}{\partial x} = -\chi'(s) \frac{\psi'(x)y}{\psi(x)^2}. \end{aligned}$$

Given a geometry F such that $f(x) = e^{-F(x)}$ satisfies

$$(9.5) \quad f(x) \lesssim \sqrt{\psi(x)},$$

we compute that $u \in W_A^{1,2}$, i.e.

$$\begin{aligned}
\int_{B(0,r)} |\nabla_A u|^2 &= \int_{B(0,r)} \left(\left| \frac{\partial u}{\partial x} \right|^2 + |f(x)|^2 \left| \frac{\partial u}{\partial y} \right|^2 \right) \\
&= \int_{B(0,r)} \left(\chi'(s)^2 \left(\frac{\psi'(x)y}{\psi(x)^2} \right)^2 + \frac{|f(x)|^2}{\psi(x)^2} \right) \\
&\lesssim \int_{B(0,r)} \chi'(s)^2 \frac{|\psi'(x)|^2 + |f(x)|^2}{\psi(x)^2} \\
&\approx \int_0^r \psi(x) \frac{|\psi'(x)|^2 + |f(x)|^2}{\psi(x)^2} \\
&\lesssim \int_0^r 1 + \psi(x) \frac{|f(x)|^2}{\psi(x)} \approx r,
\end{aligned}$$

upon using the standard inequality $\psi'(x)^2 \lesssim \psi(x)$ for smooth nonnegative ψ , and our assumption (9.5) on f .

The second derivatives of u are

$$\begin{aligned}
\frac{\partial^2}{\partial y^2} u(x, y) &= \chi''(s) \left(\frac{\partial s}{\partial y} \right)^2 + \chi'(s) \frac{\partial^2 s}{\partial y^2} = \chi''(s) \left(\frac{1}{\psi(x)} \right)^2, \\
\frac{\partial^2}{\partial x^2} u(x, y) &= \chi''(s) \left(\frac{\partial s}{\partial x} \right)^2 + \chi'(s) \frac{\partial^2 s}{\partial x^2} \\
&= \chi''(s) \left(-\frac{\psi'(x)y}{\psi(x)^2} \right)^2 + \chi'(s) \left(-\frac{\psi''(x)y}{\psi(x)^2} + \frac{2\psi(x)|\psi'(x)|^2 y}{(\psi(x)^2)^2} \right)
\end{aligned}$$

so that the operator $\mathcal{L} = \frac{\partial^2}{\partial x^2} + f(x)^2 \frac{\partial^2}{\partial y^2}$ satisfies

$$\begin{aligned}
\mathcal{L}u &= \frac{\partial^2 u}{\partial x^2} + f(x)^2 \frac{\partial^2 u}{\partial y^2} \\
&= \chi''(s) \left(-\frac{\psi'(x)y}{\psi(x)^2} \right)^2 + \chi'(s) \left(-\frac{\psi''(x)y}{\psi(x)^2} + \frac{2\psi(x)|\psi'(x)|^2 y}{(\psi(x)^2)^2} \right) \\
&\quad + f(x)^2 \chi''(s) \left(\frac{1}{\psi(x)} \right)^2 \\
&= \chi''(s) \left(\frac{|\psi'(x)|^2 y^2}{\psi(x)^4} + \frac{f(x)^2}{\psi(x)^2} \right) + \chi'(s) \left(-\frac{\psi''(x)y}{\psi(x)^2} + \frac{2|\psi'(x)|^2 y}{\psi(x)^3} \right) \\
&\equiv \chi''(s) A(x, y) + \chi'(s) B(x, y).
\end{aligned}$$

Now $\chi''(s) = \chi''\left(\frac{y}{\psi(x)}\right)$ is supported where $y \approx \psi(x)$, and $\chi'(s) = \chi'\left(\frac{y}{\psi(x)}\right)$ is supported where $y \lesssim \psi(x)$, so that

$$A(x, y) \approx \frac{|\psi'(x)|^2 + f(x)^2}{\psi(x)^2} \text{ and } |B(x, y)| \lesssim \frac{\psi''(x)}{\psi(x)} + \frac{|\psi'(x)|^2}{\psi(x)^2}.$$

Thus with

$$\phi \equiv \chi''\left(\frac{y}{\psi(x)}\right) A(x, y) + \chi'\left(\frac{y}{\psi(x)}\right) B(x, y),$$

we see that $u \in W_A^{1,2}$ is a discontinuous weak solution to the equation $\mathcal{L}u = \phi$. However we cannot expect that ϕ is A -admissible. In particular we cannot have $\mathcal{L}u \in L^{\tilde{\Phi}}$ if the strong form of the Φ -Sobolev inequality (7.15) holds.

3.1. Non-admissibility. If ψ is as above, and Φ is any Young function, we have by duality that

$$\infty = \int_0^1 \frac{dx}{x} = \int_0^1 \frac{1}{\psi(x)} \frac{1}{x} \psi(x) dx \leq \left\| \frac{1}{\psi(x)} \right\|_{L^{\Phi}(\psi(x)dx)} \left\| \frac{1}{x} \right\|_{L^{\tilde{\Phi}}(\psi(x)dx)}$$

which shows that either $\left\| \frac{1}{\psi(x)} \right\|_{L^{\Phi}(\psi(x)dx)}$ or $\left\| \frac{1}{x} \right\|_{L^{\tilde{\Phi}}(\psi(x)dx)}$ is infinite.

Now the integral arising in the endpoint inequality (7.20) (which is equivalent to the strong Φ -Sobolev inequality (7.15)) in the special case $x_1 = 0$ is essentially

$$\begin{aligned} & \int_{B(0, r_0)} \Phi(K_{B(0, r_0)}(x, y) |B(0, r_0)|) \frac{dy}{|B(0, r_0)|} \\ & \approx \int_{\Gamma_x} \Phi(h_{y_1} |B(0, r_0)|) \frac{dy_1 dy_2}{|B(0, r_0)|} \\ & \approx \int_0^{r_0} \Phi\left(\frac{1}{h_r} |B(0, r_0)|\right) h_r \frac{dr}{|B(0, r_0)|}, \end{aligned}$$

since $K(x, y) \approx \mathbf{1}_{\Gamma_x} h_{y_1 - x_1}$.

We now define ψ by $\psi(0) = 0$ and $\psi' = f$, and we wish to express the above integral in terms of ψ . We will use the estimate $h_r \approx \frac{f(r)}{|f'(r)|} = \frac{f(r)^2}{|f'(r)|^2}$ together with the following estimate,

$$(9.6) \quad \frac{f(x)^2}{|f'(x)|} \approx \psi(x).$$

It suffices to show that

$$f(x) = \frac{d}{dx} \psi(x) \approx \frac{d}{dx} \frac{f(x)^2}{f'(x)} = 2f(x) - \frac{f(x)^2 f''(x)}{f'(x)^2} = f(x) \left\{ 2 - \frac{f(x) f''(x)}{f'(x)^2} \right\}$$

for sufficiently small $x > 0$, and for this it suffices to show

$$\frac{1}{2} \leq 2 - \frac{f(x) f''(x)}{f'(x)^2} \leq 2.$$

However, the second inequality is obvious and the first is equivalent to

$$\frac{3}{2} \geq \frac{f(x) f''(x)}{f'(x)^2} = \frac{e^{-F} e^{-F} (|F'|^2 - F'')}{|e^{-F} F'|^2} = 1 - \frac{F''}{|F'|^2},$$

which is obvious since $F'' > 0$ is one of our five assumptions on the geometry F .

Thus with $x = r$ we have

$$\int_0^1 \Phi\left(\frac{1}{h_r}\right) h_r dr \approx \int_0^1 \Phi\left(\frac{|f'(x)|}{f(x)^2}\right) \frac{f(x)^2}{|f'(x)|} dx \approx \int_0^1 \Phi\left(\frac{1}{\psi(x)}\right) \psi(x) dx$$

is infinite if $\left\|\frac{1}{\psi(x)}\right\|_{L^{\Phi(\psi(x)dx)}}$ is infinite. Thus the endpoint inequality (7.20) fails, and hence the strong form of the Φ -Sobolev inequality (7.15) fails.

On the other hand, $\mathcal{L}u \approx \frac{|\psi'(x)|^2}{\psi(x)^2} \geq \frac{1}{x^2} \geq \frac{1}{x}$, and so using $h_r \approx \frac{f(r)^2}{|f'(r)|} \approx \psi(r)$ we have

$$\int_{B(0,1)} \tilde{\Phi}(\mathcal{L}u) dy_1 dy_2 \approx \int_0^1 \tilde{\Phi}\left(\frac{\psi'(x)^2}{\psi(x)^2}\right) \psi(x) dx \geq \int_0^1 \tilde{\Phi}\left(\frac{1}{x}\right) \psi(x) dx,$$

which shows that $\mathcal{L}u \notin L^{\tilde{\Phi}}$ if $\left\|\frac{1}{x}\right\|_{L^{\tilde{\Phi}(\psi(x)dx)}}$ is infinite.

CONCLUSION 104. *For the discontinuous weak solution $u(x, y) = \chi\left(\frac{y}{\psi(x)}\right)$ to the equation $\mathcal{L}u = \phi$, we must have **either** that $\phi = \mathcal{L}u \notin L^{\tilde{\Phi}}$, **or** that the strong form of the Φ -Sobolev inequality (7.15) fails. Of course it is conceivable that this function ϕ is strongly A -admissible for the geometry $f(x) = |\psi'(x)|$ using the standard form of Sobolev (7.14), or by using a different Orlicz bump Ψ altogether, but this remains unknown at this time.*

Nevertheless, using that $\frac{\partial}{\partial x} u(x, y) = \chi'\left(\frac{y}{\psi(x)}\right) \frac{y\psi'(x)}{\psi(x)^2} \approx \frac{1}{2} \frac{\psi'(x)}{\psi(x)}$ when $y = \frac{1}{2}\psi(x)$, and so that $\frac{\partial}{\partial x} u(x, y)$ is unbounded near the origin, the weak solution $v \equiv \frac{\partial}{\partial x} u(x, y)$ to $\mathcal{L}v = \phi$, as constructed above, can be used to show that the assumption of A -admissibility is *almost* necessary for all weak solutions to be locally bounded, and we show this in the subsection on the near finite type case below. On the other hand, we do not know if A -admissibility of ϕ is sufficient for all weak solutions to be locally bounded, and our main result on local boundedness requires not only that ϕ be A -admissible, but that the degeneracy of the equation be near finite type in a specific sense.

3.2. The finite type case. Suppose that $\psi(x) = \frac{1}{N+1}x^{N+1}$ with N even, and take $f(x) = \psi'(x) = x^N$. Then with $u = \chi\left(\frac{y}{\psi(x)}\right)$ we have

$$\begin{aligned} \mathcal{L}u &\lesssim \mathbf{1}_\Gamma \left(\frac{\psi''(x)}{\psi(x)} + \frac{|\psi'(x)|^2}{\psi(x)^2} \right) \\ &= \mathbf{1}_\Gamma \left(\frac{N(N-1)x^{N-2}}{x^N} + \frac{x^N}{\left(\frac{1}{N+1}x^{N+1}\right)^2} \right) \approx \mathbf{1}_\Gamma \frac{1}{x^2} \end{aligned}$$

and then since $h_r \approx r f(r) = r^{N+1}$ we have

$$\int_{B(0,1)} |\mathcal{L}u|^q \approx \int_0^1 h_r \frac{1}{r^{2q}} dr \approx \int_0^1 r^{N+1-2q} dr$$

and so $\mathcal{L}u \in L^q(B(0,1))$ if and only if $q < \frac{N+2}{2}$. But the finite type regularity theorem assumes $\Phi(t) = t^{\frac{N+2}{2}}$, so $\tilde{\Phi}(t) = t^{\frac{N+2}{2}}$ and so $q > \frac{N+2}{2}$. This shows the sharpness of the assumption $h \in L^q$ for $q > \frac{N+2}{2}$.

3.3. The near finite type case. Suppose that $\psi'(x) = e^{-(\ln \frac{1}{x})^{1+\sigma}}$ so that $\psi(x) \approx \frac{x}{(\ln \frac{1}{x})^\sigma} e^{-(\ln \frac{1}{x})^{1+\sigma}}$.

Then $\psi''(x) = -e^{-(\ln \frac{1}{x})^{1+\sigma}} (1+\sigma) (\ln \frac{1}{x})^\sigma \frac{1}{x}$ and so

$$\begin{aligned} \mathcal{L}u &\lesssim \mathbf{1}_\Gamma \left(\frac{\psi''(x)}{\psi(x)} + \frac{|\psi'(x)|^2 + f(x)^2}{\psi(x)^2} \right) \\ &\lesssim \mathbf{1}_\Gamma \left(\frac{(\ln \frac{1}{x})^{2\sigma}}{x^2} + \frac{f(x)^2 (\ln \frac{1}{x})^{2\sigma} e^{2(\ln \frac{1}{x})^{1+\sigma}}}{x^2} \right) \\ &= \mathbf{1}_\Gamma \frac{(\ln \frac{1}{x})^{2\sigma}}{x^2} \left(1 + \left[f(x) e^{(\ln \frac{1}{x})^{1+\sigma}} \right]^2 \right). \end{aligned}$$

Note that if we take $f(x) = \psi'(x) = e^{-(\ln \frac{1}{x})^{1+\sigma}}$, then

$$\mathcal{L}u \lesssim \mathbf{1}_\Gamma \frac{(\ln \frac{1}{x})^{2\sigma}}{x^2},$$

while if we take $f(x) = \sqrt{\psi(x)}$, then

$$\mathcal{L}u \lesssim \mathbf{1}_\Gamma \frac{(\ln \frac{1}{x})^\sigma}{x} e^{(\ln \frac{1}{x})^{1+\sigma}} = \mathbf{1}_\Gamma \frac{1}{\psi(x)}.$$

Consider a near power bump $\Phi_m(t) = e^{((\ln t)^{\frac{1}{m}} + 1)^m}$. Now we compute $\widetilde{\Phi_m}$:

$$\begin{aligned} \Phi'_m(t) &= e^{((\ln t)^{\frac{1}{m}} + 1)^m} \left\{ m \left((\ln t)^{\frac{1}{m}} + 1 \right)^{m-1} \frac{1}{m} (\ln t)^{\frac{1}{m}-1} \frac{1}{t} \right\} \\ &= e^{((\ln t)^{\frac{1}{m}} + 1)^m} \left((\ln t)^{\frac{1}{m}} + 1 \right)^{m-1} (\ln t)^{-\frac{m-1}{m}} \frac{1}{t}. \end{aligned}$$

Now take the inverse of $s = \Phi'_m(t)$:

$$\begin{aligned} \ln s &= \left((\ln t)^{\frac{1}{m}} + 1 \right)^m + (m-1) \ln \left((\ln t)^{\frac{1}{m}} + 1 \right) - \frac{m-1}{m} \ln \ln t - \ln t \\ &= \left((\ln t)^{\frac{1}{m}} + 1 \right)^m \left\{ 1 + \frac{(m-1) \ln \left((\ln t)^{\frac{1}{m}} + 1 \right) - \frac{m-1}{m} \ln \ln t - \ln t}{\left((\ln t)^{\frac{1}{m}} + 1 \right)^m} \right\} \\ &\approx \left(\ln t + m (\ln t)^{\frac{m-1}{m}} \right) \left\{ 1 - \frac{\ln t}{\ln t + m (\ln t)^{\frac{m-1}{m}}} \right\} = m (\ln t)^{\frac{m-1}{m}}, \end{aligned}$$

which implies that

$$\ln t \approx \left(\frac{1}{m} \ln s \right)^{\frac{m}{m-1}}.$$

Thus

$$\begin{aligned} \widetilde{\Phi_m}'(s) &= [\Phi'_m]^{-1}(s) = t \approx e^{(\frac{1}{m} \ln s)^{\frac{m}{m-1}}}; \\ \widetilde{\Phi_m}(s) &\approx \frac{s}{\frac{1}{m-1} (\frac{1}{m} \ln s)^{\frac{1}{m-1}}} e^{(\frac{1}{m} \ln s)^{\frac{m}{m-1}}} \approx \frac{e^{\ln s + (\frac{1}{m} \ln s)^{\frac{m}{m-1}}}}{(\ln s)^{\frac{1}{m-1}}}. \end{aligned}$$

Now we compute to see when $\mathcal{L}u \in L^{\widehat{\Phi}_m}$. Assuming that $f(x) = \psi'(x)$, we have

$$\ln \mathcal{L}u = \ln \frac{(\ln \frac{1}{x})^{2\sigma}}{x^2} = 2\sigma \ln \ln \frac{1}{x} + 2 \ln \frac{1}{x},$$

and so

$$\begin{aligned} & \int_{B(0,1)} \widehat{\Phi}_m(\mathcal{L}u) \\ & \lesssim \int_{B(0,1)} \frac{e^{\ln \mathcal{L}u + (\frac{1}{m} \ln \mathcal{L}u)^{\frac{m}{m-1}}}}{(\ln \mathcal{L}u)^{\frac{1}{m-1}}} dx dy \leq \int_0^1 \psi(x) \frac{e^{\ln \mathcal{L}u + (\frac{1}{m} \ln \mathcal{L}u)^{\frac{m}{m-1}}}}{(\ln \mathcal{L}u)^{\frac{1}{m-1}}} dx \\ & = \int_0^1 \frac{\exp\left\{-\ln \frac{1}{x} - (\ln \frac{1}{x})^{1+\sigma}\right\}}{(\ln \frac{1}{x})^\sigma} \frac{\exp\left\{2\sigma \ln \ln \frac{1}{x} + 2 \ln \frac{1}{x} + \left(\frac{1}{m} (2\sigma \ln \ln \frac{1}{x} + 2 \ln \frac{1}{x})\right)^{\frac{m}{m-1}}\right\}}{(2\sigma \ln \ln \frac{1}{x} + 2 \ln \frac{1}{x})^{\frac{1}{m-1}}} dx \\ & \leq \int_0^1 \frac{\exp\left\{-\ln \frac{1}{x} - (\ln \frac{1}{x})^{1+\sigma} + 2\sigma \ln \ln \frac{1}{x} + 2 \ln \frac{1}{x} + \left(\frac{1}{m} (2\sigma \ln \ln \frac{1}{x} + 2 \ln \frac{1}{x})\right)^{\frac{m}{m-1}}\right\}}{(\ln \frac{1}{x})^\sigma (2\sigma \ln \ln \frac{1}{x} + 2 \ln \frac{1}{x})^{\frac{1}{m-1}}} dx \\ & = \int_0^1 \frac{\exp\left\{-\left(\ln \frac{1}{x}\right)^{1+\sigma} + 2\sigma \ln \ln \frac{1}{x} + \ln \frac{1}{x} + \frac{1}{m^{\frac{m}{m-1}}} (2\sigma \ln \ln \frac{1}{x} + 2 \ln \frac{1}{x})^{\frac{m}{m-1}}\right\}}{(\ln \frac{1}{x})^\sigma (2\sigma \ln \ln \frac{1}{x} + 2 \ln \frac{1}{x})^{\frac{1}{m-1}}} dx. \end{aligned}$$

Now in the event that $1 + \sigma > \frac{m}{m-1}$, i.e. $\sigma > \frac{1}{m-1}$, then the numerator is clearly bounded and so the integral is finite. But we only have a Φ_m -Sobolev inequality for the geometry F_σ if $\sigma < \frac{1}{m-1}$.

Finally we compute

$$\begin{aligned} \widehat{\phi}(x, y) & \equiv \mathcal{L} \frac{\partial}{\partial x} u = \left\{ \frac{\partial^2}{\partial x^2} + \psi'(x)^2 \frac{\partial^2}{\partial y^2} \right\} \frac{\partial}{\partial x} \chi\left(\frac{y}{\psi(x)}\right) \\ & = \frac{\partial}{\partial x} \left\{ \frac{\partial^2}{\partial x^2} + \psi'(x)^2 \frac{\partial^2}{\partial y^2} \right\} \chi\left(\frac{y}{\psi(x)}\right) - 2\psi'(x) \psi''(x) \frac{\partial^2}{\partial y^2} \chi\left(\frac{y}{\psi(x)}\right) \\ & = O\left(\left|\frac{\psi'(x)}{\psi(x)}\right|^3 + \left|\frac{\psi'(x)}{\psi(x)}\right| \left|\frac{\psi''(x)}{\psi(x)}\right| + \left|\frac{\psi'''(x)}{\psi(x)}\right|\right). \end{aligned}$$

CONCLUSION 105. Let $\frac{1}{m-1} < \sigma < \frac{1}{m'-1}$. Then $\phi \equiv \mathcal{L}u \in L^{\widehat{\Phi}_m}$ and the $\Phi_{m'}$ -Sobolev inequality (7.15) holds. Thus u is a discontinuous weak solution to the equation $\mathcal{L}u = \phi$, where ϕ comes arbitrarily close to being strongly A -admissible for the geometry F_σ in the sense that $|m - m'|$ can be made as small as we wish. Moreover, $\frac{\partial}{\partial x} u$ is a locally unbounded weak solution to the equation $\mathcal{L} \frac{\partial}{\partial x} u = \widehat{\phi}$, where $\widehat{\phi}$ also comes arbitrarily close to being strongly A -admissible in the sense that $\widehat{\phi} \in L^{\widehat{\Phi}_m}$ and the $\Phi_{m'}$ -Sobolev inequality (7.15) holds, and where $|m - m'|$ can be made as small as we wish.

In particular, the above conclusion shows that if all nonnegative weak subsolutions u to $\mathcal{L}u = \phi \in L^{\widehat{\Phi}}$ are locally bounded, then the Φ -Sobolev inequality holds.

An extension of the theory to three dimensions

In this second chapter of the fourth part of the paper, we consider an extension of the theory above to the model operator

$$\mathcal{L}_1 \equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + f(x_1)^2 \frac{\partial^2}{\partial x_3^2},$$

of Kusuoka and Strook [KusStr] who have shown (see also M. Christ [Chr] and the references given there for a nice survey of the linear situation) that when $f(x_1)$ is smooth and positive away from the origin, the operator \mathcal{L}_1 is hypoelliptic if and only if

$$\lim_{r \rightarrow 0} r \ln f(r) = 0.$$

We consider the analogous problems for local boundedness and continuity of appropriate weak solutions to rough divergence form operators $L_1 = \operatorname{div} \mathcal{A} \nabla$ with quadratic forms \mathcal{A} controlled by that of \mathcal{L}_1 . In particular, we show that

- for our geometries in the range where \mathcal{L}_1 fails to be hypoelliptic, our operator L_1 fails to be weakly hypoelliptic - in fact there are unbounded weak solutions u to the homogeneous equation $L_1 u = 0$, and
- for our geometries in the range where \mathcal{L}_1 is hypoelliptic, we establish local boundedness and maximum principles for all weak subsolutions u to admissible equations $L_1 u = \phi$, but only provided the degeneracy of f is an entire log better than $\lim_{r \rightarrow 0} r \ln f(r) = 0$. This gap arises as a consequence of the failure of Moser iteration in the absence of an Orlicz Sobolev inequality for a bump function Φ_m with $m > 2$. It remains an open question whether or not local boundedness and the maximum principle hold for subsolutions u to $L_1 u = \phi$ for the geometries in this gap.

1. The Kusuoka-Strook operator \mathcal{L}_1

We first compute the geodesics and areas of metric balls corresponding to the operator \mathcal{L}_1 , and then use this to calculate the corresponding subrepresentation inequality. Then we compute the Orlicz bump Sobolev norms and obtain local boundedness and continuity of weak solutions. Finally, we show that for very degenerate geometries, there exist unbounded weak solutions u to the homogeneous equation $\mathcal{L}_1 u = 0$.

1.1. Geodesics and metric balls. Let $\gamma(t) = (x_1(t), x_2(t), x_3(t))$ be a path. Then the arc length element is given by

$$ds = \sqrt{[x'_1(t)]^2 + [x'_2(t)]^2 + \frac{1}{[f(x_1)]^2} [x'_3(t)]^2} dt.$$

Thus we can factor the associated control space by

$$\left(\mathbb{R}^3, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & [f(x_1)]^{-2} \end{bmatrix} \right) = \left(\mathbb{R}_{x_1, x_3}^2, \begin{bmatrix} 1 & 0 \\ 0 & [f(x_1)]^{-2} \end{bmatrix} \right) \times \mathbb{R}_{x_2}.$$

We begin with a lemma regarding paths in product spaces.

LEMMA 106. *Let (M_1, g^{M_1}) and (M_2, g^{M_2}) be two Riemannian manifolds. Let us consider the Cartesian product $M_1 \times M_2$ whose Riemann product is defined by*

$$g_{(p,q)}((u_1, u_2), (v_1, v_2)) = g_p^{M_1}(u_1, v_1) + g_q^{M_2}(u_2, v_2).$$

Here

$$(p, q) \in M_1 \times M_2 \text{ and } (u_1, u_2), (v_1, v_2) \in T_p(M_1) \oplus T_q(M_2) \approx T_{(p,q)}(M_1 \times M_2).$$

Given any C^1 path $\gamma : [a, b] \rightarrow M_1 \times M_2$, we can write it in the form $(\gamma_1(t), \gamma_2(t))$, here $\gamma_1 : [a, b] \rightarrow M_1$ and $\gamma_2 : [a, b] \rightarrow M_2$ are C^1 paths on M_1 and M_2 , respectively. Then we have

$$\|\gamma\| \geq \sqrt{\|\gamma_1\|^2 + \|\gamma_2\|^2}$$

Where $\|\gamma\|$, $\|\gamma_1\|$ and $\|\gamma_2\|$ represent the arc length of each path. In addition, the equality happens if and only if

$$(10.1) \quad \frac{\|\gamma'_1(t)\|_{g^{M_1}}}{\|\gamma_1\|} = \frac{\|\gamma'_2(t)\|_{g^{M_2}}}{\|\gamma_2\|}, \quad a \leq t \leq b.$$

PROOF. For simplicity we omit the subscripts of the norms $\|\gamma'_1(t)\|_{g^{M_1}}$ and $\|\gamma'_2(t)\|_{g^{M_2}}$ so that $\|\gamma_j\| = \int_a^b \sqrt{\|\gamma'_j(t)\|^2} dt$. Using that

$$\begin{aligned} \frac{\|\gamma_1\|}{\sqrt{\|\gamma_1\|^2 + \|\gamma_2\|^2}} \|\gamma'_1(t)\| + \frac{\|\gamma_2\|}{\sqrt{\|\gamma_1\|^2 + \|\gamma_2\|^2}} \|\gamma'_2(t)\| \\ \leq \sqrt{\|\gamma'_1(t)\|^2 + \|\gamma'_2(t)\|^2}, \end{aligned}$$

with equality if and only if

$$\begin{pmatrix} \|\gamma'_1(t)\| \\ \|\gamma'_2(t)\| \end{pmatrix} \text{ is parallel to } \begin{pmatrix} \|\gamma_1\| \\ \|\gamma_2\| \end{pmatrix},$$

we obtain that

$$\begin{aligned} \|\gamma\| &= \int_a^b \sqrt{\|\gamma'_1(t)\|^2 + \|\gamma'_2(t)\|^2} dt \\ &\geq \int_a^b \left(\frac{\|\gamma_1\|}{\sqrt{\|\gamma_1\|^2 + \|\gamma_2\|^2}} \|\gamma'_1(t)\| + \frac{\|\gamma_2\|}{\sqrt{\|\gamma_1\|^2 + \|\gamma_2\|^2}} \|\gamma'_2(t)\| \right) dt \\ &= \frac{\|\gamma_1\|^2}{\sqrt{\|\gamma_1\|^2 + \|\gamma_2\|^2}} + \frac{\|\gamma_2\|^2}{\sqrt{\|\gamma_1\|^2 + \|\gamma_2\|^2}} = \sqrt{\|\gamma_1\|^2 + \|\gamma_2\|^2}, \end{aligned}$$

with equality if and only if (10.1) holds. ■

COROLLARY 107. *A C^1 path $\gamma = (\gamma_1, \gamma_2)$ is a geodesic of $M_1 \times M_2$ if and only if*

(1) γ_1 is a geodesic of M_1 ,

- (2) γ_2 is a geodesic of M_2 ,
 (3) and the speeds of γ_1 and γ_2 match, i.e. the identity $\frac{\|\gamma_1'(t)\|_{g^{M_1}}}{\|\gamma_1\|} = \frac{\|\gamma_2'(t)\|_{g^{M_2}}}{\|\gamma_2\|}$ holds for all t .

COROLLARY 108. The distance between two points $(p_1, q_1), (p_2, q_2) \in M_1 \times M_2$ is given by

$$d_g((p_1, q_1), (p_2, q_2)) = \sqrt{[d_{g^{M_1}}(p_1, p_2)]^2 + [d_{g^{M_2}}(q_1, q_2)]^2}.$$

Thus we can write a typical geodesic in the form

$$\begin{cases} x_2 = C_2 \pm k \int_0^{x_1} \frac{\lambda}{\sqrt{\lambda^2 - [f(u)]^2}} du \\ x_3 = C_3 \pm \int_0^{x_1} \frac{[f(u)]^2}{\sqrt{\lambda^2 - [f(u)]^2}} du \end{cases},$$

and a metric ball centered at $y = (y_1, y_2, y_3)$ with radius $r > 0$ is given by

$$B(y, r) \equiv \left\{ (x_1, x_2, x_3) : (x_1, x_3) \in B_{2D} \left((y_1, y_3), \sqrt{r^2 - |x_2 - y_2|^2} \right) \right\},$$

where $B_{2D}(a, s)$ denotes the 2-dimensional control ball centered at a in the plane with radius t that was associated with f above.

1.1.1. *Volumes of three dimensional balls.* Recall that the Lebesgue measure of the *two* dimensional ball $B_{2D}(x, r)$ satisfies

$$|B_{2D}(x, r)| \approx \begin{cases} r^2 f(x_1) & \text{if } r \leq \frac{1}{|F'(x_1)|} \\ \frac{f(x_1+r)}{|F'(x_1+r)|^2} & \text{if } r \geq \frac{1}{|F'(x_1)|} \end{cases}.$$

LEMMA 109. The Lebesgue measure of the three dimensional ball $B(x, r)$ satisfies

$$\begin{aligned} |B(x, r)| &\approx \begin{cases} r^3 f(x_1) & \text{if } r \leq \frac{1}{|F'(x_1)|} \\ \frac{f(x_1+r)}{|F'(x_1+r)|^3} & \text{if } r \geq \frac{1}{|F'(x_1)|} \end{cases} \\ &\approx |B_{2D}(x, r)| \min \left\{ r, \frac{1}{|F'(x_1)|} \right\}. \end{aligned}$$

Thus to pass from areas of two dimensional balls to volumes of three dimensional balls, we simply multiply the area of the ball by the factor $\min \left\{ r, \frac{1}{|F'(x_1)|} \right\}$.

PROOF. To estimate the measure $|B(x, r)|$ of a *three* dimensional ball $B(x, r)$ we can assume without loss of generality that $x_2 = x_3 = 0$ and we then consider two cases.

Case $r < \frac{1}{|F'(x_1)|}$: In this case we have $\sqrt{r^2 - y_2^2} \leq r < \frac{1}{|F'(x_1)|}$ and

$$\left| B_{2D} \left((x_1, x_3), \sqrt{r^2 - y_2^2} \right) \right| \approx (r^2 - y_2^2) f(x_1),$$

which gives

$$|B(x, r)| = \int_0^r \left| B_{2D} \left((x_1, x_3), \sqrt{r^2 - y_2^2} \right) \right| dy_2 \approx \int_0^r (r^2 - y_2^2) f(x_1) dy_2 \approx r^3 f(x_1).$$

Case $r \geq \frac{1}{|F'(x_1)|}$: In this case the integral in $y_2 \in (0, r)$ is divided into two regions.

Region 1: $0 < \sqrt{r^2 - y_2^2} \leq \frac{1}{|F'(x_1)|}$. In this region we have $\sqrt{r^2 - \frac{1}{|F'(x_1)|^2}} \leq y_2 \leq r$ and

$$\left| B_{2D} \left((x_1, x_3), \sqrt{r^2 - y_2^2} \right) \right| \approx (r^2 - y_2^2) f(x_1).$$

Thus we obtain

$$\begin{aligned} \int_{\sqrt{r^2 - \frac{1}{|F'(x_1)|^2}}}^r \left| B_{2D} \left((x_1, x_3), \sqrt{r^2 - y_2^2} \right) \right| dy_2 &\approx \int_{\sqrt{r^2 - \frac{1}{|F'(x_1)|^2}}}^r (r^2 - y_2^2) f(x_1) dy_2 \\ &= \left(r^2 \left(r - \sqrt{r^2 - \frac{1}{|F'(x_1)|^2}} \right) - \frac{1}{3} \left(r^3 - \left(\sqrt{r^2 - \frac{1}{|F'(x_1)|^2}} \right)^3 \right) \right) f(x_1) \\ &\approx \frac{rf(x_1)}{|F'(x_1)|^2}, \end{aligned}$$

where we used the estimate

$$r - \sqrt{r^2 - \frac{1}{|F'(x_1)|^2}} = \frac{\frac{1}{|F'(x_1)|^2}}{r + \sqrt{r^2 - \frac{1}{|F'(x_1)|^2}}} \approx \frac{1}{r|F'(x_1)|^2}.$$

Region 2: $\frac{1}{|F'(x_1)|} \leq \sqrt{r^2 - y_2^2} \leq r$. In this region we have $0 \leq y_2 \leq \sqrt{r^2 - \frac{1}{|F'(x_1)|^2}}$ and

$$\left| B_{2D} \left((x_1, x_3), \sqrt{r^2 - y_2^2} \right) \right| \approx \frac{f \left(x_1 + \sqrt{r^2 - y_2^2} \right)}{\left| F' \left(x_1 + \sqrt{r^2 - y_2^2} \right) \right|^2}.$$

Thus

$$\int_0^{\sqrt{r^2 - \frac{1}{|F'(x_1)|^2}}} \left| B_{2D} \left((x_1, x_3), \sqrt{r^2 - y_2^2} \right) \right| dy_2 \approx \frac{f(x_1 + r)}{|F'(x_1 + r)|^3},$$

where we used

$$\left(\frac{f(s)}{|F'(s)|^2} \right)' \approx \frac{f(s)}{|F'(s)|}$$

so for $\delta = \frac{1}{2|F'(x_1 + r)|}$, we have

$$\frac{f(x_1 + r - \delta)}{|F'(x_1 + r - \delta)|^2} \approx \delta \left(\frac{f(s)}{|F'(s)|^2} \right)' \Big|_{s=x_1+r} = \frac{1}{2} \frac{f(x_1 + r)}{|F'(x_1 + r)|^2}$$

by the tangent line approximation.

Combining the estimates for Regions 1 and 2 we obtain for the case $r \geq \frac{1}{|F'(x_1)|}$ that

$$|B(x, r)| \approx \frac{f(x_1 + r)}{|F'(x_1 + r)|^3}.$$

■

1.2. Subrepresentation inequalities. The subrepresentation inequality here is similar to Lemma 84 in two dimensions, with the main difference being in the definition of the cusp-like region $\Gamma(x, r)$ in three dimensions. In three dimensions we define

$$\Gamma(x, r) = \bigcup_{k=1}^{\infty} E(x, r_k);$$

$$E(x, r_k) \equiv \left\{ y = (y_1, y_2, y_3) : \begin{array}{l} x_1 + r_{k+1} \leq y_1 < x_1 + r_k \\ |y_2| < \sqrt{r^2 - (y_1 - x_1)^2} \\ |y_3| < h^*(x_1, y_1 - x_1) \end{array} \right\},$$

where just as in the two dimensional case, we can show $|E(x, r_k)| \approx |E(x, r_k) \cap B(x, r_k)| \approx |B(x, r_k)|$. Let $|B(x, d(x, y))|$ denote the three dimensional Lebesgue measure of $B(x, d(x, y))$ where $d(x, y)$ is now the three dimensional control distance.

LEMMA 110. *With notation as above, in particular $r_0 = r$, r_1 given by (7.1), and assuming $\int_{E(x, r_1)} w = 0$, we have the subrepresentation formula*

$$w(x) \leq C \int_{\Gamma(x, r)} |\nabla_A w(y)| \frac{\widehat{d}(x, y)}{|B(x, d(x, y))|} dy,$$

where ∇_A is as in (1.10) and

$$\widehat{d}(x, y) \equiv \min \left\{ d(x, y), \frac{1}{|F'(x_1 + d(x, y))|} \right\}.$$

The proof is very similar to that of the two dimensional analogue, Lemma 84 above, and is left to the reader.

1.3. Sobolev Orlicz bump inequalities. Let $T_{B(0, r_0)}$ be the positive integral operator with kernel $K_{B(0, r_0)}$ defined as in (7.6),

$$K_{B(0, r_0)}(x, y) \equiv \frac{\widehat{d}(x, y)}{|B(x, d(x, y))|} \mathbf{1}_{\Gamma(x, r_0)}(y),$$

and recall the *strong* (Φ, φ) -Sobolev Orlicz bump inequality (7.15),

$$(10.2) \quad \Phi^{(-1)} \left(\int_{B(0, r_0)} \Phi(T_{B(0, r_0)} g) d\mu \right) \leq C\varphi(r_0) \|g\|_{L^1(\mu_{r_0})}.$$

Note: Define the dilate Φ_δ of Φ by $\Phi_\delta(t) = \delta\Phi(\frac{t}{\delta})$. Then the above strong Φ -Sobolev Orlicz bump inequality holds for Φ if and only if it holds for all dilates Φ_δ . Indeed, with $s = \Phi_\delta(t)$ we have $\Phi_\delta^{(-1)}(s) = t = \delta\Phi^{(-1)}(\frac{s}{\delta})$, and so (10.2) implies

$$\begin{aligned} \Phi_\delta^{(-1)} \left(\int_{B(0, r_0)} \Phi_\delta(T_{B(0, r_0)} g) d\mu \right) &= \delta\Phi^{(-1)} \left(\frac{1}{\delta} \int_{B(0, r_0)} \delta\Phi \left(T_{B(0, r_0)} \frac{g}{\delta} \right) d\mu \right) \\ &= \delta\Phi^{(-1)} \left(\int_{B(0, r_0)} \Phi \left(T_{B(0, r_0)} \frac{g}{\delta} \right) d\mu \right) \\ &\leq \delta C\varphi(r_0) \left\| \frac{g}{\delta} \right\|_{L^1(\mu_{r_0})} = C\varphi(r_0) \|g\|_{L^1(\mu_{r_0})}. \end{aligned}$$

We have the following three dimensional version of Proposition 92, where by a geometry F , we now mean the three dimensional geometry with metric

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & f(x_1)^2 \end{bmatrix}, \quad f(s) = e^{-F(s)}.$$

PROPOSITION 111. *Let $0 < r_0 < 1$ and $C_m > 0$. Suppose that the geometry F satisfies the monotonicity property:*

$$(10.3) \quad \varphi(r) \equiv \frac{1}{|F'(r)|} e^{C_m \left(\frac{|F'(r)|^2}{F''(r)} + 1 \right)^{m-1}} \text{ is an increasing function of } r \in (0, r_0).$$

Then the (Φ, φ) -Sobolev inequality (7.15) holds with geometry F , with φ as in (7.21) and with Φ as in (7.19), $m > 1$.

The analogue of Corollary 93 holds here as well, and its proof is essentially the same as before.

COROLLARY 112. *The strong Φ -Sobolev inequality (10.2) with Φ as in (7.19), $m > 1$, and geometry $F = F_{k,\sigma}$ holds if
(either) $k \geq 2$ and $\sigma > 0$ and $\varphi(r_0)$ is given by*

$$\varphi(r_0) = r_0^{1 - C_m \frac{\left(\ln^{(k)} \frac{1}{r_0} \right)^{\sigma(m-1)}}{\ln \frac{1}{r_0}}}, \quad \text{for } 0 < r_0 \leq \beta_{m,\sigma},$$

for a positive constants C_m and $\beta_{m,\sigma}$ depending only on m and σ ;

(or) $k = 1$ and $\sigma < \frac{1}{m-1}$ and $\varphi(r_0)$ is given by

$$\varphi(r_0) = r_0^{1 - C_m \frac{1}{\left(\ln \frac{1}{r_0} \right)^{1 - \sigma(m-1)}}}, \quad \text{for } 0 < r_0 \leq \beta_{m,\sigma},$$

for positive constants C_m and $\beta_{m,\sigma}$ depending only on m and σ .

Conversely, the standard (Φ, φ) -Sobolev inequality (7.14) with Φ as in (7.19), $m > 1$, fails if $k = 1$ and $\sigma > \frac{1}{m-1}$.

REMARK 113. Recall that in the two dimensional case, we had

$$|B_{2D}(x, d(x, y))| \approx h_{x,y} \widehat{d}(x, y).$$

In the three dimensional case, the quantities $h_{x,y}$ and $\widehat{d}(x, y)$ remain formally the same and

$$\begin{aligned} |B(x, d(x, y))| &\approx |B_{2D}(x, d(x, y))| \min \left\{ d(x, y), \frac{1}{|F'(x_1 + d(x, y))|} \right\} \\ &= |B_{2D}(x, d(x, y))| \widehat{d}(x, y). \end{aligned}$$

Thus in three dimensions we have

$$|B(x, d(x, y))| \approx h_{x,y} \widehat{d}(x, y)^2,$$

and hence the estimate,

$$\begin{aligned} K_B(x, y) &= \frac{\hat{d}(x, y)}{|B(x, d(x, y))|} \mathbf{1}_{\Gamma(x, r_0)}(y) \\ &\approx \frac{1}{\hat{d}(x, y) h_{y_1 - x_1}} \approx \begin{cases} \frac{1}{r^2 f(x_1)}, & 0 < r = y_1 - x_1 < \frac{1}{|F'(x_1)|} \\ \frac{|F'(x_1 + r)|^2}{f(x_1 + r)}, & 0 < r = y_1 - x_1 \geq \frac{1}{|F'(x_1)|} \end{cases}. \end{aligned}$$

Thus the three dimensional kernel $K_B(x, y)$ is obtained from the corresponding two dimensional kernel by dividing by the factor $\hat{d}(x, y)$. On the other hand, the volume $|B(x, d(x, y))|$ of the three dimensional ball $B(x, d(x, y))$ is obtained from the corresponding area of the two dimensional ball by multiplying by the factor $\hat{d}(x, y)$. This has **roughly** the same effect as replacing the bump function $\Phi(t)$ with the dilate $\Phi_\delta(t)$ where $\delta = \hat{d}(x, y)$. By the Note preceding Proposition 111 we expect that (10.2) holds for a three dimensional geometry F if and only if (7.15) holds for the corresponding two dimensional geometry F . Of course $\delta = \hat{d}(x, y)$ is not a constant and so below we carefully modify the proof of Proposition 92 by adjusting for the factor $\frac{1}{d(x, y)}$ in three dimensional kernel, and the factor $\hat{d}(x, y)$ in the volume of the three dimensional ball.

PROOF OF PROPOSITION 111. Just as in the proof of Proposition 92, it suffices to prove the analogue of (7.23), i.e.

$$\int_B \Phi \left(\frac{K(x, y)|B|}{\omega(r(B))} \right) d\mu(y) \leq C_m \varphi(r(B)) |F'(r(B))|,$$

for all small balls B of radius $r(B)$ centered at the origin, and where $\omega(r(B))$ is the same as in the proof of Proposition 92, i.e.

$$\omega(r(B)) = \frac{1}{t_m |F'(r(B))|}, \quad t_m > e^{2^m}.$$

Here $|B|$ and $K(x, y)$ are now given by

$$|B(0, r_0)| \approx \frac{f(r_0)}{|F'(r_0)|^3},$$

and

$$K(x, y) \equiv \frac{1}{s_{y_1 - x_1}} \approx \begin{cases} \frac{1}{r^2 f(x_1)}, & 0 < r = y_1 - x_1 < \frac{1}{|F'(x_1)|} \\ \frac{|F'(x_1 + r)|^2}{f(x_1 + r)}, & 0 < r = y_1 - x_1 \geq \frac{1}{|F'(x_1)|} \end{cases},$$

where we are writing $\frac{1}{K(x, y)}$ as $s_{y_1 - x_1} = s_r$ since the quantity s_r is essentially a cross sectional area analogous to the height h_r in the two dimensional case. As before, write $\Phi(t)$ as

$$\Phi(t) = t^{1+\psi(t)}, \quad \text{for } t > 0,$$

where for $t \geq E$,

$$\psi(t) = \left(1 + (\ln t)^{-\frac{1}{m}}\right)^m - 1 \approx \frac{m}{(\ln t)^{1/m}},$$

and for $t < E$,

$$\psi(t) = \frac{\ln \frac{\Phi(E)}{E}}{\ln t}.$$

Then arguing just as before it suffices to prove the analogue of (7.25),

$$(10.4) \quad \mathcal{I}_{0, r_0 - x_1} = \frac{1}{\omega(r_0)} \int_0^{r_0 - x_1} \left(\frac{|B(0, r_0)|}{s_r \omega(r_0)} \right)^{\psi\left(\frac{|B(0, r_0)|}{s_r \omega(r_0)}\right)} dr \leq C_m \varphi(r_0) |F'(r_0)|,$$

where C_0 is a sufficiently large positive constant, and of course $|B(0, r_0)|$ is now the Lebesgue measure of the three dimensional ball $B(0, r_0)$.

To prove this we divide the interval $(0, r_0 - x_1)$ of integration in r into three regions as before:

(1): the small region \mathcal{S} where $\frac{|B(0, r_0)|}{s_r \omega(r_0)} \leq E$,

(2): the big region \mathcal{R}_1 that is disjoint from \mathcal{S} and where $r = y_1 - x_1 < \frac{1}{|F'(x_1)|}$ and

(3): the big region \mathcal{R}_2 that is disjoint from \mathcal{S} and where $r = y_1 - x_1 \geq \frac{1}{|F'(x_1)|}$.

The region \mathcal{S} is handled just as before.

We now turn to the first big region \mathcal{R}_1 where we have $s_{y_1 - x_1} \approx r^2 f(x_1)$. The condition that \mathcal{R}_1 is disjoint from \mathcal{S} gives

$$\frac{|B(0, r_0)|}{r^2 f(x_1) \omega(r_0)} > E, \quad \text{i.e. } r < \sqrt{\frac{A}{E}};$$

$$\text{where } A = A(x_1) \equiv \frac{|B(0, r_0)|}{f(x_1) \omega(r_0)},$$

and so as before

$$\int_{\mathcal{R}_1} \Phi \left(K_{B(0, r_0)}(x, y) \frac{|B(0, r_0)|}{\omega(r_0)} \right) \frac{dy}{|B(0, r_0)|} = \frac{1}{\omega(r_0)} \int_0^{\min\left\{\sqrt{\frac{A}{E}}, \frac{1}{|F'(x_1)|}\right\}} \left(\frac{A}{r^2} \right)^{\psi\left(\frac{A}{r^2}\right)} dr.$$

We now claim the analogue of (7.26),

$$\frac{1}{\omega(r_0)} \int_0^{\min\left\{\sqrt{\frac{A}{E}}, \frac{1}{|F'(x_1)|}\right\}} \left(\frac{A}{r^2} \right)^{\psi\left(\frac{A}{r^2}\right)} dr \lesssim \Phi(t_m),$$

Now if $\sqrt{\frac{A}{E}} \leq \frac{1}{|F'(x_1)|}$, then with the change of variable $t = \frac{A}{r^2}$,

$$\begin{aligned} & \frac{1}{\omega(r_0)} \int_0^{\min\left\{\sqrt{\frac{A}{E}}, \frac{1}{|F'(x_1)|}\right\}} \left(\frac{A}{r^2} \right)^{\psi\left(\frac{A}{r^2}\right)} dr = C \frac{1}{\omega(r_0)} \sqrt{A} \int_E^\infty t^{\psi(t)} \frac{dt}{t^{\frac{3}{2}}} \\ & \leq \frac{1}{\omega(r_0)} \sqrt{A} C_\varepsilon \int_E^\infty t^{\varepsilon - \frac{3}{2}} dt = C_\varepsilon \frac{1}{\omega(r_0)} \sqrt{A} \leq C_\varepsilon t_m r_0 |F'(r_0)|, \end{aligned}$$

which proves (10.4) if $\sqrt{\frac{A}{E}} \leq \frac{1}{|F'(x_1)|}$ since $r_0 \leq \varphi(r_0)$.

So we now suppose that $\sqrt{\frac{A}{E}} > \frac{1}{|F'(x_1)|}$. Making a change of variables

$$R = \frac{A}{r^2} = \frac{A(x_1)}{r^2},$$

we obtain

$$\frac{1}{\omega(r_0)} \int_0^{\frac{1}{|F'(x_1)|}} \left(\frac{A}{r^2} \right)^{\psi\left(\frac{A}{r^2}\right)} dr = \frac{1}{\omega(r_0)} \sqrt{A} \int_{A|F'(x_1)|^2}^{\infty} R^{\psi(R)-\frac{3}{2}} dR.$$

Integrating by parts gives as before

$$\begin{aligned} \int_{A|F'(x_1)|^2}^{\infty} R^{\psi(R)-\frac{3}{2}} dR &= \int_{A|F'(x_1)|^2}^{\infty} R^{\psi(R)+1} \left(-\frac{2}{3} \frac{1}{R^{\frac{3}{2}}} \right)' dR \\ &\leq \frac{2}{3} \frac{(A|F'(x_1)|^2)^{\psi(A|F'(x_1)|^2)}}{\sqrt{A|F'(x_1)|^2}} + \frac{2}{3} \left(1 + \frac{m-1}{(\ln E)^{\frac{1}{m}}} \right) \int_{A|F'(x_1)|^2}^{\infty} R^{\psi(R)-\frac{3}{2}} dR \end{aligned}$$

Taking E large enough depending on m we can assure

$$\frac{2}{3} \left(1 + \frac{m-1}{(\ln E)^{\frac{1}{m}}} \right) \leq \frac{3}{4},$$

which gives

$$\int_{A|F'(x_1)|^2}^{\infty} R^{\psi(R)-\frac{3}{2}} dR \lesssim \frac{(A|F'(x_1)|^2)^{\psi(A|F'(x_1)|^2)}}{\sqrt{A|F'(x_1)|^2}},$$

and therefore

$$\begin{aligned} \mathcal{I}_{0, \frac{1}{|F'(x_1)|}}(x) &= \frac{1}{\omega(r_0)} \sqrt{A} \int_{A|F'(x_1)|^2}^{\infty} R^{\psi(R)-\frac{3}{2}} dR \\ &\lesssim \frac{1}{\omega(r_0) |F'(x_1)|} (A(x_1) |F'(x_1)|^2)^{\psi(A(x_1) |F'(x_1)|^2)} \equiv \frac{1}{\omega(r_0)} \mathcal{F}(x_1); \\ c &= f(x_1) A(x_1) = \frac{f(r_0)}{\omega(r_0) |F'(r_0)|^3} = \frac{t_m f(r_0)}{|F'(r_0)|^2}. \end{aligned}$$

We now look for the maximum of the function $\mathcal{F}(x_1)$ given by

$$\mathcal{F}(x_1) \equiv (A(x_1) |F'(x_1)|^2)^{\psi(A(x_1) |F'(x_1)|^2)} = \frac{1}{|F'(x_1)|} \left(c(r_0) \frac{|F'(x_1)|^2}{f(x_1)} \right)^{\psi\left(c(r_0) \frac{|F'(x_1)|^2}{f(x_1)}\right)}$$

where

$$c(r_0) = \frac{t_m f(r_0)}{|F'(r_0)|^2}.$$

Note that this expression is very similar to the function $\mathcal{F}(x_1)$ defined in the proof of Proposition 92 except for $|F'(x_1)|$ being squared in the argument of the exponential and a multiplication by a constant. It is also the same function that was maximized in the proof of Proposition 97 so using that result we have

$$\mathcal{F}(x_1) \leq \frac{1}{|F'(x_1^*)|} e^{C_m \left(1 + \frac{|F'(x_1^*)|^2}{F''(x_1^*)} \right)^{m-1}},$$

where $x_1^* \in (0, r_0)$ is the value of x_1 which maximizes $\mathcal{F}(x_1)$. By monotonicity property (10.3) we thus conclude

$$\mathcal{F}(x_1) \leq \varphi(r_0),$$

and therefore

$$\mathcal{I}_{0, \frac{1}{|F'(x_1)|}}(x) \leq C_m \varphi(r_0) |F'(r_0)|$$

which is (10.4) for the region \mathcal{R}_1 .

For the second big region \mathcal{R}_2 we have

$$\frac{1}{s_{y_1-x_1}} \approx \frac{|F'(x_1+r)|^2}{f(x_1+r)},$$

and the integral to be estimated becomes

$$I_{\mathcal{R}_2} \equiv \frac{1}{\omega(r_0)} \int_{x_1 + \frac{1}{|F'(x_1)|}}^{r_0} \left(\frac{f(r_0)|F'(y_1)|^2}{f(y_1)|F'(r_0)|^3 \omega(r_0)} \right)^{\psi \left(\frac{f(r_0)|F'(y_1)|^2}{f(y_1)|F'(r_0)|^3 \omega(r_0)} \right)} dy_1.$$

This integral is again similar to the integral $I_{\mathcal{R}_2}$ from the proof of Proposition 92 except for $|F'(y_1)|$ being squared in the integrand. We leave it to the reader to verify that the same analysis gives the desired estimate for $I_{\mathcal{R}_2}$ in this case. ■

1.4. Generalized Inner Ball inequality. Here we consider the generalized Inner Ball inequality (1.24), namely

$$(10.5) \quad \|u\|_{L^\infty(\nu B_r)} \leq C_r e^{c(\ln \frac{1}{1-\nu})^m} \|u\|_{L^2(B_r)},$$

for the control balls B_r associated with the geometry for \mathcal{L}_1 when $f(x_1) = x_1^{\left(\ln \frac{1}{x_1}\right)^\sigma}$ for the equation

$$(10.6) \quad \mathcal{L}_1^\sigma u = \phi, \quad \text{where } \phi \text{ is admissible,}$$

with $\sigma > 0$ and $m > 1$. We will show that for $0 < \sigma < 1$, (10.5) holds for $m = 1 + \sigma$, and that this choice of m is optimal.

To see this, we modify a standard idea, as presented in [Chr], for establishing necessary conditions for hypoellipticity. Our modification consists in comparing L^∞ and L^2 norms for a certain Schrödinger operator \mathcal{L}_1^τ defined below. This will result in demonstrating sharpness of the growth parameter m in the generalized Inner Ball inequality. We first rewrite our operator \mathcal{L}_1 in variables (x, y, t) as

$$\mathcal{L}_1 = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - f(x)^2 \frac{\partial^2}{\partial t^2}.$$

For each $\tau > 0$ consider the one-dimensional Schrödinger operator

$$\mathcal{L}_1^\tau \equiv -\frac{\partial^2}{\partial x^2} + \tau^2 f(x)^2$$

on $L^2(\mathbb{R})$. Since the potential $\tau f(x)^2$ is positive away from the origin, it has a discrete spectrum tending to ∞ , and its least eigenvalue $\lambda_0^2(\tau)$ satisfies

$$\lambda_0^2(\tau) = \sup_{g \neq 0} \frac{\langle \mathcal{L}_1^\tau g, g \rangle}{\langle g, g \rangle} = \sup_{g \neq 0} \frac{\int_{\mathbb{R}} \left(\tau^2 f(x)^2 |g(x)|^2 + |g'(x)|^2 \right) dx}{\int_{\mathbb{R}} |g(x)|^2 dx}.$$

We normalize the corresponding eigenvector g_τ so that $g_\tau(0) = 1$. Then g_τ is even, strictly positive, and achieves its maximum value of 1 at $x = 0 \in \mathbb{R}$.

Now we consider the function

$$G_\tau(x, y, t) \equiv e^{i\tau t} e^{\lambda_0(\tau)y} g_\tau(x).$$

Then it is easy to check that

$$G_\tau(x, y, t) \equiv f_\tau(x) e^{\lambda_0(\tau)y} e^{i\tau t}$$

is a solution to

$$\mathcal{L}_1 G_\tau = 0$$

since

$$\begin{aligned} \mathcal{L}_1 G_\tau(x, y, t) &= e^{i\tau t} e^{\lambda_0(\tau)y} \left\{ -g_\tau''(x) - \lambda_0^2(\tau) g_\tau(x) + \tau^2 f(x)^2 g_\tau(x) \right\} \\ &= e^{i\tau t} e^{\lambda_0(\tau)y} \left\{ \mathcal{L}_1^\tau g_\tau(x) - \lambda_0^2(\tau) g_\tau(x) \right\} = 0 \end{aligned}$$

since $\mathcal{L}_1^\tau g_\tau = \lambda_0^2(\tau) g_\tau$.

Now suppose we have a generalized Inner Ball inequality which we can write in the form

$$(10.7) \quad \|u\|_{L^\infty(\nu B)} \leq C_r e^{C(\ln \frac{1}{1-\nu})^m} \|u\|_{L^2(B; \mu_r)}$$

for u , any solution of $Lu = \phi$ with ϕ admissible. Therefore, we can substitute $u = F_\tau$ into (10.7). For the left hand side this immediately gives

$$(10.8) \quad \|G_\tau\|_{L^\infty(\nu B)} = e^{\lambda_0(\tau)\nu r}$$

For the RHS we first estimate the L^2 -norm

$$(10.9) \quad \|G_\tau\|_{L^2(\mu_r)}^2 = \int g_\tau(x)^2 e^{2\lambda_0(\tau)y} \frac{dt dx dy}{|B(0, r)|} \leq \int_{-r}^r e^{2\lambda_0(\tau)y} \frac{|B_{2D}(0, \sqrt{r^2 - y^2})|}{|B(0, r)|} dy$$

where B_{2D} denotes a 2-dimensional ball on the (x, t) plane. To estimate the integral we look for local maxima of the integrand since it is equal to zero at the endpoints. First, recall that $|B_{2D}(0, R)|$ is given by

$$|B_{2D}(0, R)| \approx \frac{f(R)}{|F'(R)|^2}$$

which in the case of F_σ geometry translates to

$$|B_{2D}(0, R)| \approx \frac{R^2 e^{-(\ln \frac{1}{R})^{1+\sigma}}}{(\ln \frac{1}{R})^2}$$

Differentiating $e^{2\lambda_0(\tau)y} f(\sqrt{r^2 - y^2})/|F'(\sqrt{r^2 - y^2})|^2$ with respect to y and putting to 0 we obtain

$$2\lambda_0(\tau) = \frac{y|F'(\sqrt{r^2 - y^2})|}{\sqrt{r^2 - y^2}} \left(1 - \frac{2F''(\sqrt{r^2 - y^2})}{|F'(\sqrt{r^2 - y^2})|^2} \right)$$

Applying this to F_σ geometry this gives

$$\lambda_0(\tau) \approx \frac{y}{r^2 - y^2} \left(\ln \frac{1}{\sqrt{r^2 - y^2}} \right)^\sigma \leq \frac{y}{(r^2 - y^2)^{1+\varepsilon}}$$

where the last inequality is true for any $\varepsilon > 0$ and small enough $r > 0$. We thus have the following implicit estimate on y^* that maximizes the integrand in (10.9)

$$r^2 - y^{*2} \lesssim \left(\frac{y^*}{\lambda_0(\tau)} \right)^{\frac{1}{1+\varepsilon}}$$

Substituting this back to (10.9) gives

$$\begin{aligned} \|G_\tau\|_{L^2(\mu_r)}^2 &\leq \frac{Cr}{|B(0, r)|} \left| B_{2D} \left(0, \left(\frac{y^*}{\lambda_0(\tau)} \right)^{\frac{1}{2+2\varepsilon}} \right) \right| e^{2\lambda_0(\tau)y^*} \leq \frac{Cr}{|B(0, r)|} \left| B_{2D} \left(0, \left(\frac{r}{\lambda_0(\tau)} \right)^{\frac{1}{2+2\varepsilon}} \right) \right| e^{2\lambda_0(\tau)r} \\ &\leq C_r e^{2\lambda_0(\tau)r} \frac{e^{-C(\ln \lambda_0(\tau))^{1+\sigma}}}{\lambda_0(\tau)^{\frac{1}{1+\varepsilon}} (\ln \lambda_0(\tau))^{2\sigma}} \end{aligned}$$

where in the last inequality we used the explicit expression for F_σ and combined all the terms depending only on r and not on λ_0 in C_r . Together with (10.7) and (10.8) this implies

$$e^{\lambda_0(\tau)\nu r} \leq C_r e^{C(\ln \frac{1}{1-\nu})^m} e^{\lambda_0(\tau)r} \frac{e^{-C(\ln \lambda_0(\tau))^{1+\sigma}}}{\lambda_0(\tau)^{\frac{1}{2+2\varepsilon}} (\ln \lambda_0(\tau))^\sigma}$$

or dividing through by $e^{\lambda_0(\tau)\nu r}$

$$1 \leq C_r e^{C(\ln \frac{1}{1-\nu})^m} e^{\lambda_0(\tau)(1-\nu)r} \frac{e^{-C(\ln \lambda_0(\tau))^{1+\sigma}}}{\lambda_0(\tau)^{\frac{1}{1+\varepsilon}} (\ln \lambda_0(\tau))^\sigma}$$

The above inequality should hold for all $0 < \nu_0 \leq \nu < 1$, therefore, since $\lambda_0(\tau) \rightarrow \infty$ as $\tau \rightarrow \infty$ we can choose ν such that

$$\frac{1}{1-\nu} = \lambda_0(\tau)$$

Substituting in the above inequality we have

$$1 \leq C_r \frac{e^{C(\ln \lambda_0(\tau))^m} e^{-C(\ln \lambda_0(\tau))^{1+\sigma}}}{\lambda_0(\tau)^{\frac{1}{1+\varepsilon}} (\ln \lambda_0(\tau))^\sigma}$$

To satisfy this for all $\lambda_0(\tau)$ we must require $m > \sigma + 1$.

On the other hand, it is easy to see that the generalized Inner Ball inequality (10.7) holds for $m = 1 + \sigma$ whenever we have the Inner Ball inequality just for the choice $\nu = \frac{1}{2}$, i.e.

$$(10.10) \quad \|u\|_{L^\infty(\frac{1}{2}B_r)} \leq C_r \|u\|_{L^2(B_r; \mu_r)}, \quad \text{for all balls } B_r \text{ of radius } 0 < r < R.$$

Indeed, to see this, fix $\frac{1}{2} < \nu < 1$ and a point $P \in \nu B_r$. Then the ball $B(P, (1-\nu)r)$ is contained in B_r , and so from (10.10) applied to the ball $B(P, (1-\nu)r)$ we obtain

$$u(P) \leq \|u\|_{L^\infty(\frac{1}{2}B(P, (1-\nu)r))} \leq C_r \|u\|_{L^2(B(P, (1-\nu)r); \mu_r)}.$$

Now we note that

$$\begin{aligned} \|u\|_{L^2(B(P, (1-\nu)r); \mu_r)}^2 &= \frac{1}{|B(P, (1-\nu)r)|} \int_{B(P, (1-\nu)r)} |u|^2 \\ &\leq \frac{|B_r|}{|B(P, (1-\nu)r)|} \frac{1}{|B_r|} \int_{B_r} |u|^2 \\ &\leq \frac{|B_r|}{|B_{(1-\nu)r}|} \frac{1}{|B_r|} \int_{B_r} |u|^2 \\ &\approx C_r e^{(\ln \frac{1}{1-\nu})^{1+\sigma}} \frac{1}{|B_r|} \int_{B_r} |u|^2 \\ &= C_r e^{(\ln \frac{1}{1-\nu})^{1+\sigma}} \|u\|_{L^2(B_r; \mu_r)}^2, \end{aligned}$$

which proves the generalized Inner Ball inequality (10.7), equivalently (10.5), holds for $m = 1 + \sigma$.

From the above analyses we can conclude the following for all $\sigma > 0$.

- If $\sigma < 1$ we can find $m > 2$ such that $\sigma < \frac{1}{m-1}$, and therefore (10.10) holds, and hence also (10.7) for $m = 1 + \sigma$, for all admissible right hand sides.
- If $\sigma > 0$ and $m > \sigma + 1$, then (10.7) fails.

From these two bullet items we conclude that for $0 < \sigma < 1$, the generalized Inner Ball inequality (10.7) holds for $m = 1 + \sigma$, and for no larger value of m .

1.5. Local boundedness and continuity of weak solutions. Using Proposition 111, we can now extend Theorems 16 and 17 to the three dimensional operator \mathcal{L}_1 , and using an analogous version of Proposition 97 for three dimensions, whose formulation and proof we leave for the reader, we can extend Theorem 26 to the three dimensional operator \mathcal{L}_1 . The proofs of these extensions of Theorems 16, 17 and 26 follow the arguments in the two-dimensional case treated earlier, using the three dimensional analogues just discussed above. This finally completes the proofs of all of the theorems stated in the introduction.

THEOREM 114. *Suppose that u is a weak solution to the infinitely degenerate equation $L_1 u \equiv \nabla^{\text{tr}} \mathcal{A} \nabla u = \phi$ in $\Omega \subset \mathbb{R}^3$, where the matrix \mathcal{A} satisfies (1.9) and ϕ is A -admissible, and where the degeneracy function f in (1.9) is comparable to $f_{k,\sigma}$.*

(1) *If*

$$\text{either } k \geq 2 \text{ and } \sigma > 0; \text{ or } k = 1 \text{ and } 0 < \sigma < 1,$$

then u is locally bounded in Ω , i.e. for all compact subsets K of Ω ,

$$\|u\|_{L^\infty(K)} \leq C'_K \left(1 + \|u\|_{L^2(\Omega)}\right),$$

and satisfies the maximum principle

$$\sup_{\Omega} \leq \sup_{\partial\Omega} + \|\phi\|_{X(\Omega)}.$$

(2) *If in addition ϕ is Dini A -admissible and the degeneracy function f in (1.9) satisfies (1.2), then u is continuous in Ω .*

2. An unbounded weak solution

In this final section of Part 3, we demonstrate that weak solutions to our degenerate equations can fail to be locally bounded. We modify an example of Morimoto [Mor] that was used to provide an alternate proof of a result of Kusuoka and Strook [KuStr].

THEOREM 115. *Suppose that $g \in C^\infty(\mathbb{R})$ satisfies $g(x) \geq 0$, $g(0) = 0$ and the decay condition*

$$(10.11) \quad \liminf_{x \rightarrow 0} \left| x \ln \frac{1}{g(x)} \right| \neq 0.$$

Then for some $\varepsilon > 0$, the operator

$$\mathcal{L} \equiv \frac{\partial^2}{\partial x^2} + g(x) \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial t^2}$$

fails to be $W_A^{1,2}(\mathbb{R}^2)$ -hypoelliptic in an open subset $(-1, 1) \times \mathbb{R} \times (-\varepsilon, \varepsilon)$ of \mathbb{R}^3 containing the origin, where $\nabla_A \equiv \left(\frac{\partial}{\partial x}, \sqrt{g(x)} \frac{\partial}{\partial y}, \frac{\partial}{\partial t} \right)$ is the degenerate gradient associated with \mathcal{L} .

PROOF. For $a, \eta > 0$ we follow Morimoto [Mor], who in turn followed Bouendi and Goulaouic [BoGo], by considering the second order operator $L_\eta \equiv -\frac{\partial^2}{\partial x^2} + g(x)\eta^2$ and the eigenvalue problem

$$\begin{aligned} L_\eta v(x, \eta) &= \lambda v(x, \eta), & x \in I_a \equiv (-a, a), \\ v(a, \eta) &= v(-a, \eta) = 0. \end{aligned}$$

The least eigenvalue is given by the Rayleigh quotient formula

$$\begin{aligned} \lambda_0(a, \eta) &= \inf_{f(\neq 0) \in C_0^\infty(I_a)} \frac{\langle L_\eta f, f \rangle_{L^2}}{\|f\|_{L^2}^2} \\ &= \inf_{f(\neq 0) \in C_0^\infty(I_a)} \frac{\int_{-a}^a f'(x)^2 dx + \int_{-a}^a g(x)\eta^2 f(x)^2 dx}{\|f\|_{L^2}^2}, \end{aligned}$$

from which it follows that

$$(10.12) \quad \lambda_0(a, \eta) \leq \lambda_0(a_0, \eta) \text{ if } a \geq a_0.$$

The decay condition (10.11) above is equivalent to the existence of $\delta_0 > 0$ such that $g(x) \leq e^{-\frac{\delta_0}{|x|}}$ for x small. So we may suppose $g(x) \leq Ce^{-\frac{\delta_0}{|x|}}$ for $x \in I \equiv [-1, 1]$ where $C \geq 1$, and then take $|\eta|$ sufficiently large that with

$$a(\eta) \equiv \frac{\delta_0}{\ln C + 2 \ln |\eta|},$$

we have both $a(\eta) \leq 1$ and

$$g(x)\eta^2 \leq Ce^{-\frac{\delta_0}{|x|}}\eta^2 \leq e^{\ln C - \frac{\delta_0}{a(\eta)} + 2 \ln |\eta|} = 1, \quad x \in I_{a(\eta)}.$$

Now let $\mu_0(a(\eta))$ denote the least eigenvalue for the problem

$$\begin{aligned} \left\{ -\frac{\partial^2}{\partial x^2} + 1 \right\} v(x, \eta) &= \mu v(x, \eta), & x \in I_{a(\eta)} = (-a(\eta), a(\eta)), \\ v(a(\eta), \eta) &= v(-a(\eta), \eta) = 0, \end{aligned}$$

and note that

$$\begin{aligned} \mu_0(a(\eta)) &= \inf_{f(\neq 0) \in C_0^\infty(I_{a(\eta)})} \frac{\left\langle -\frac{\partial^2 f}{\partial x^2} + f, f \right\rangle_{L^2(I_{a(\eta)})}}{\|f\|_{L^2(I_{a(\eta)})}^2} \\ &= \inf_{f(\neq 0) \in C_0^\infty(I_{a(\eta)})} \frac{\int_{-a(\eta)}^{a(\eta)} f'(x)^2 dx + \int_{-a(\eta)}^{a(\eta)} f(x)^2 dx}{\|f\|_{L^2(I_{a(\eta)})}^2}. \end{aligned}$$

It follows that

$$\lambda_0(a(\eta), \eta) \leq \mu_0(a(\eta)), \quad \text{for } |\eta| \text{ sufficiently large.}$$

Now an easy classical calculation using exact solutions to $\left\{ -\frac{\partial^2}{\partial x^2} + 1 - \mu \right\} v = 0$ shows that

$$\mu_0(a(\eta)) = C_1 \frac{1}{a(\eta)^2} + 1,$$

for some constant C_1 independent of η , and hence combining this with (10.12), we have

$$(10.13) \quad \begin{aligned} 0 &< \lambda_0(1, \eta) \leq \lambda_0(a(\eta), \eta) \leq \mu_0(a(\eta)) \\ &= C_1 \left(\frac{\ln C + 2 \ln |\eta|}{\delta_0} \right)^2 + 1 \leq C_2 (\ln |\eta|)^2, \quad \text{for } |\eta| \text{ sufficiently large.} \end{aligned}$$

Now let $v_0(x, \eta)$ be an eigenfunction on the interval $I = -I_1 = [-1, 1]$ associated with $\lambda_0(1, \eta)$ and normalized so that

$$(10.14) \quad \|v_0(\cdot, \eta)\|_{L^2(I)} = 1.$$

Choose a sequence $\{a_n\}_{n=-\infty}^{\infty}$ satisfying

$$(10.15) \quad |a_n| \leq \frac{1}{1 + n^\alpha} \rho_n,$$

for some $\alpha > 0$ where $\|\{\rho_n\}_{n \in \mathbb{Z}}\|_{\ell^2} = 1$. For $y \in I_\pi = [-\pi, \pi]$, identified with the unit circle \mathbb{T} upon identifying $-\pi$ and π , we formally define

$$\begin{aligned} w(x, y) &\equiv \sum_{n \in \mathbb{Z}} e^{iyn} v_0(x, n) a_n; \\ w_N(x, y) &\equiv \sum_{n \in \mathbb{Z}} \lambda_0(1, n)^N e^{iyn} v_0(x, n) a_n; \\ u(x, y, t) &\equiv \sum_{N=0}^{\infty} \frac{t^{2N}}{(2N)!} w_N(x, y). \end{aligned}$$

We now claim that

$$w_N(x, y) = \left\{ -\frac{\partial^2}{\partial x^2} - g(x) \frac{\partial^2}{\partial y^2} \right\}^N w(x, y).$$

Indeed, assuming this holds for N , and using

$$-\frac{\partial^2}{\partial x^2} v_0(x, n) = [\lambda_0(1, n) - g(x) n^2] v_0(x, n),$$

we obtain that

$$\begin{aligned}
\left\{ -\frac{\partial^2}{\partial x^2} - g(x) \frac{\partial^2}{\partial y^2} \right\}^{N+1} w(x, y) &= \left\{ -\frac{\partial^2}{\partial x^2} - g(x) \frac{\partial^2}{\partial y^2} \right\} w_N(x, y) \\
&= - \sum_{n \in \mathbb{Z}} \lambda_0(1, n)^N e^{iyn} \frac{\partial^2}{\partial x^2} v_0(x, n) a_n \\
&\quad + \sum_{n \in \mathbb{Z}} \lambda_0(1, n)^N e^{iyn} g(x) n^2 v_0(x, n) a_n \\
&= \sum_{n \in \mathbb{Z}} \lambda_0(1, n)^N e^{iyn} \lambda_0(1, n) v_0(x, n) a_n \\
&\quad - \sum_{n \in \mathbb{Z}} \lambda_0(1, n)^N e^{iyn} g(x) n^2 v_0(x, n) a_n \\
&\quad + \sum_{n \in \mathbb{Z}} \lambda_0(1, n)^N e^{iyn} g(x) n^2 v_0(x, n) a_n \\
&= \sum_{n \in \mathbb{Z}} \lambda_0(1, n)^{N+1} e^{iyn} v_0(x, n) a_n = w_{N+1}(x, y).
\end{aligned}$$

It follows that

$$\left\{ -\frac{\partial^2}{\partial x^2} - g(x) \frac{\partial^2}{\partial y^2} \right\} w_N(x, y) = w_{N+1}(x, y),$$

and so formally we get

$$\begin{aligned}
\mathcal{L}u(x, y, t) &= \left\{ -\frac{\partial^2}{\partial x^2} - g(x) \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial t^2} \right\} \sum_{N=0}^{\infty} \frac{t^{2N}}{(2N)!} w_N(x, y) \\
&= \sum_{N=0}^{\infty} \frac{t^{2N}}{(2N)!} w_{N+1}(x, y) - \sum_{N=0}^{\infty} \frac{\partial^2}{\partial t^2} \frac{t^{2N}}{(2N)!} w_N(x, y) \\
&= \sum_{N=0}^{\infty} \frac{t^{2N}}{(2N)!} w_{N+1}(x, y) - \sum_{N=1}^{\infty} \frac{t^{2N-2}}{(2N-2)!} w_N(x, y) = 0.
\end{aligned}$$

Now we show that $u(x, y, t)$ is well defined as an $L^2(I \times \mathbb{T})$ -valued analytic function of t for t in some small neighbourhood of 0 provided $\{a_n\}_{n \in \mathbb{Z}}$ is in $\ell^2(\mathbb{Z})$ with suitable decay at ∞ , namely (10.15). Indeed, using Plancherel's formula in the y variable, and then Fubini's theorem, we have

$$\begin{aligned}
\|w_N\|_{L^2(I \times \mathbb{T})}^2 &= \int_{-1}^1 \left\{ \int_{-\pi}^{\pi} \left| \sum_{n \in \mathbb{Z}} \lambda_0(1, n)^N e^{iyn} v_0(x, n) a_n \right|^2 dy \right\} dx \\
&= \int_{-1}^1 \left\{ \sum_{n \in \mathbb{Z}} \left| \lambda_0(1, n)^N v_0(x, n) a_n \right|^2 \right\} dx \\
&= \sum_{n \in \mathbb{Z}} \left\{ \int_{-1}^1 |v_0(x, n)|^2 dx \right\} \left| \lambda_0(1, n)^N a_n \right|^2 \\
&= \sum_{n \in \mathbb{Z}} \left| \lambda_0(1, n)^N a_n \right|^2.
\end{aligned}$$

Now from (10.13) we have the bound $\lambda_0(1, n) \leq C_2 (\ln n)^2$ for n sufficiently large, and hence from (10.15),

$$|a_n| \leq \frac{1}{1+n^\alpha} \rho_n,$$

where $\|\{\rho_n\}_{n \in \mathbb{Z}}\|_{\ell^2} = 1$, we obtain

$$\begin{aligned} \|w_N\|_{L^2(I \times \mathbb{T})} &\leq C_3 \sqrt{\sum_{n \in \mathbb{Z}} |(\ln n)^{2N} a_n|^2} \\ &\leq C_3 \sqrt{\sum_{n \in \mathbb{Z}} \left| (\ln n)^{2N} \left(\frac{1}{1+e^{\alpha \ln n}} \right) \right|^2 |\rho_n|^2} \\ &\leq C_4 \sqrt{N} \alpha^{-2N} (2N)! \sqrt{\sum_{n \in \mathbb{Z}} |\rho_n|^2} = C_4 \sqrt{N} \alpha^{-2N} (2N)! \end{aligned}$$

since by Stirling's formula,

$$\frac{s^{2N}}{1+e^{\alpha s}} \leq s^{2N} e^{-\alpha s} \leq \left(\frac{2N}{\alpha} \right)^{2N} e^{-\alpha \frac{2N}{\alpha}} = \left(\frac{2N}{e} \right)^{2N} \alpha^{-2N} \leq \sqrt{N} \alpha^{-2N} (2N)!$$

Thus we conclude that

$$\|u(x, y, t)\|_{L^2_{x,y}(I \times \mathbb{T})} \leq \sum_{N=0}^{\infty} \frac{t^{2N}}{(2N)!} \|w_N\|_{L^2(I \times \mathbb{T})} \leq C_4 \sum_{N=0}^{\infty} \sqrt{N} \left(\frac{t}{\alpha} \right)^{2N} < \infty$$

for $t \in (-\alpha, \alpha)$, and it follows that $u(x, y, t)$ is a well-defined $L^2(I \times \mathbb{T})$ -valued analytic function of $t \in (-\alpha, \alpha)$ that satisfies the homogeneous equation $\mathcal{L}u(x, y, t) = 0$ for $(x, y, t) \in I \times \mathbb{T} \times (-\alpha, \alpha)$.

Next, we show that $u \in W_A^{1,2}(I \times \mathbb{T} \times (-\varepsilon, \varepsilon))$ for some $\varepsilon > 0$. We first compute that

$$\begin{aligned} &\left\| \frac{\partial}{\partial x} w_N \right\|_{L^2(I \times \mathbb{T})}^2 + \left\| \sqrt{g(x)} \frac{\partial}{\partial y} w_N \right\|_{L^2(I \times \mathbb{T})}^2 \\ &= \left\langle \left\{ -\frac{\partial^2}{\partial x^2} - g(x) \frac{\partial^2}{\partial y^2} \right\} w_N(x, y), w_N(x, y) \right\rangle_{L^2(I \times \mathbb{T})} \\ &= \langle w_{N+1}(x, y), w_N(x, y) \rangle_{L^2(I \times \mathbb{T})} \\ &\leq \|w_{N+1}\|_{L^2(I \times \mathbb{T})} \|w_N\|_{L^2(I \times \mathbb{T})} \\ &\leq C_4 \sqrt{N+1} \alpha^{-2N-2} (2N+2)! C_4 \sqrt{N} \alpha^{-2N} (2N)! \\ &\leq C_5^2 N^3 [(2N)!]^2 \alpha^{-4N-2}, \end{aligned}$$

which shows in particular that $w_N \in W_A^{1,2}(I \times \mathbb{T})$ for each $N \geq 1$ with the norm estimate

$$\left\| \frac{w_N}{(2N)!} \right\|_{W_A^{1,2}(I \times \mathbb{T})} \leq C_5 N^{\frac{3}{2}} \alpha^{-2N-1}.$$

Thus the $W_A^{1,2}(I \times \mathbb{T})$ -valued analytic function $u(t) = \sum_{N=0}^{\infty} \frac{w_N}{(2N)!} t^{2N}$ is $W_A^{1,2}(I \times \mathbb{T})$ -bounded in the complex disk $B(0, \alpha)$ centered at the origin with radius α . Then we use Cauchy's estimates for the $W_A^{1,2}(I \times \mathbb{T})$ -valued analytic function $u(t)$ to obtain that $\frac{\partial}{\partial t} u(t)$ is $W_A^{1,2}(I \times \mathbb{T})$ -bounded in any complex disk $B(0, \varepsilon)$ with $0 < \varepsilon < \alpha = \beta - \frac{1}{2}$, which shows that $\frac{\partial}{\partial t} u \in L^2(I \times \mathbb{T} \times (-\varepsilon, \varepsilon))$ for $0 < \varepsilon < \beta - \frac{1}{2}$. This completes the proof that $u \in W_A^{1,2}(I \times \mathbb{T} \times (-\varepsilon, \varepsilon))$ for some $\varepsilon > 0$.

Finally, we note that with $\rho_n = \frac{1}{1+|n|^\beta}$ where $\frac{1}{2} < \beta \leq \frac{3}{2} - \alpha$, then u is *not* smooth near the origin since

$$\begin{aligned} \left\| \frac{\partial}{\partial y} w \right\|_{L^2(I \times \mathbb{T})}^2 &= \int_{-1}^1 \left\{ \int_{-\pi}^{\pi} \left| \sum_{n \in \mathbb{Z}} i n e^{i y n} v_0(x, n) a_n \right|^2 dy \right\} dx \\ &= \int_{-1}^1 \left\{ \sum_{n \in \mathbb{Z}} |v_0(x, n) n a_n|^2 \right\} dx \\ &= \sum_{n \in \mathbb{Z}} \left\{ \int_{-1}^1 |v_0(x, n)|^2 dx \right\} |n a_n|^2 \\ &= \sum_{n \in \mathbb{Z}} |n a_n|^2 = \sum_{n \in \mathbb{Z}} \left| \frac{n}{1+|n|^\alpha} \rho_n \right|^2 = \infty. \end{aligned}$$

This is essentially the example of Morimoto [Mor]. However, we need more - namely, we must construct an unbounded weak solution u in some neighbourhood of the origin in $I \times \mathbb{T} \times (-\varepsilon, \varepsilon)$.

To accomplish this, we first derive the additional property (10.16) below of the least eigenfunction $v_n(x) \equiv v_0(x, n)$ that satisfies the equation

$$\begin{aligned} \left\{ -\frac{\partial^2}{\partial x^2} + g(x) n^2 \right\} v_n(x) &= \lambda_0(1, n) v_n(x), \\ v_n(-1) &= v_n(1) = 0. \end{aligned}$$

We claim that $v_n(x)$ is even on $[-1, 1]$ and decreasing from $v_n(0)$ to 0 on the interval $[0, 1]$. Indeed, the least eigenfunction v_n minimizes the Rayleigh quotient

$$\frac{\int_{-1}^1 v_n'(x)^2 dx + \int_{-1}^1 g(x) \eta^2 v_n(x)^2 dx}{\|v_n\|_{L^2}^2} = \inf_{f(\neq 0) \in C_0^\infty(I_a)} \frac{\int_{-1}^1 f'(x)^2 dx + \int_{-1}^1 g(x) \eta^2 f(x)^2 dx}{\|f\|_{L^2}^2},$$

and since the radially decreasing rearrangement v_n^* of v_n on $[-1, 1]$ satisfies both

$$\int_{-1}^1 v_n^{*'}(x)^2 dx \leq \int_{-1}^1 v_n'(x)^2 dx \text{ and } \int_{-1}^1 g(x) \eta^2 v_n^*(x)^2 dx \leq \int_{-1}^1 g(x) \eta^2 f(x)^2 dx,$$

as well as $\|v_n^*\|_{L^2}^2 = \|v_n\|_{L^2}^2$, we conclude that $v_n = v_n^*$. The only simple consequence we need from this is that

$$(10.16) \quad 2v_n(0)^2 \geq \int_{-1}^1 v_n(x)^2 dx = 1, \quad n \geq 1,$$

where the equality follows from our normalizing assumption $\|v_n\|_{L^2} = 1$ in (10.14).

Now recall $\alpha > 0$ from (10.15) above, and choose $0 < \alpha < \alpha' \leq \frac{1}{4}$ and define

$$(10.17) \quad a_n = \begin{cases} \frac{1}{n^{\frac{1}{2} + \alpha'}} & \text{for } n \geq 1 \\ 0 & \text{for } n \leq 0 \end{cases}.$$

Then for each $x \in I$, we have with $b_n(x) \equiv v_n(x) a_n$,

$$\begin{aligned} w(x, y) &= \sum_{n=1}^{\infty} e^{iy_n} v_n(x) a_n = \sum_{n=1}^{\infty} e^{iy_n} b_n(x) , \\ w(x, y)^2 &= \left(\sum_{n=1}^{\infty} e^{iy_n} b_n(x) \right)^2 = \sum_{n=2}^{\infty} \left\{ \sum_{k=1}^{n-1} b_{n-k}(x) b_k(x) \right\} e^{iy_n} , \end{aligned}$$

and so by Plancherel's theorem,

$$\|w(x, \cdot)\|_{L^4(\mathbb{T})}^4 = \|w(x, \cdot)^2\|_{L^2(\mathbb{T})}^2 = \sum_{n=2}^{\infty} \left| \sum_{k=1}^{n-1} b_{n-k}(x) b_k(x) \right|^2 .$$

In particular we have from (10.16) that

$$\begin{aligned} \|w(0, \cdot)\|_{L^4(\mathbb{T})}^4 &= \sum_{n=2}^{\infty} \left| \sum_{k=1}^{n-1} b_{n-k}(0) b_k(0) \right|^2 \\ &= \sum_{n=2}^{\infty} \left| \sum_{k=1}^{n-1} v_{n-k}(0) a_{n-k} v_k(0) a_k \right|^2 \geq \frac{1}{2} \sum_{n=2}^{\infty} \left| \sum_{k=1}^{n-1} a_{n-k} a_k \right|^2 , \end{aligned}$$

and now we obtain that $\|w(0, \cdot)\|_{L^4(\mathbb{T})}^4 = \infty$ from the estimates

$$\begin{aligned} \sum_{k=1}^{n-1} a_{n-k} a_k &= \sum_{k=1}^{n-1} \frac{1}{(n-k)^{\frac{1}{2}+\alpha'}} \frac{1}{k^{\frac{1}{2}+\alpha'}} \gtrsim \frac{1}{(n-\frac{n}{2})^{\frac{1}{2}+\alpha'}} \sum_{k=1}^{\frac{n}{2}} \frac{1}{k^{\frac{1}{2}+\alpha'}} \\ &\gtrsim \frac{1}{(\frac{n}{2})^{\frac{1}{2}+\alpha'}} \left(\frac{n}{2}\right)^{\frac{1}{2}-\alpha'} \approx \frac{1}{n^{2\alpha'}} \end{aligned}$$

and

$$\|w(0, \cdot)\|_{L^4(\mathbb{T})}^4 \geq \frac{1}{2} \sum_{n=2}^{\infty} \left| \sum_{k=1}^{n-1} a_{n-k} a_k \right|^2 \gtrsim \sum_{n=2}^{\infty} \frac{1}{n^{4\alpha'}} = \infty, \quad \text{for } \alpha' \leq \frac{1}{4}.$$

Now we note that each eigenfunction $v_n(x)$ is continuous in x since it solves an elliptic second order equation on the interval $[-1, 1]$. Then we write

$$\begin{aligned} \sum_{n=2}^{\infty} \left| \sum_{k=1}^{n-1} v_{n-k}(x) a_{n-k} v_k(x) a_k \right|^2 &= \sum_{n=2}^{\infty} \left| \sum_{k=1}^{n-1} b_{n-k}(x) b_k(x) \right|^2 \\ &= \sum_{\substack{\beta=(\beta_1, \beta_2, \beta_3, \beta_4) \in \mathbb{N}^4 \\ \beta_1 + \beta_2 = n = \beta_3 + \beta_4}} b_{\beta_1}(x) b_{\beta_2}(x) b_{\beta_3}(x) b_{\beta_4}(x) , \end{aligned}$$

and apply Fatou's lemma to conclude that

$$\begin{aligned}
\infty &= \sum_{n=2}^{\infty} \left| \sum_{k=1}^{n-1} b_{n-k}(0) b_k(0) \right|^2 = \sum_{\substack{\beta=(\beta_1, \beta_2, \beta_3, \beta_4) \in \mathbb{N}^4 \\ \beta_1 + \beta_2 = n = \beta_3 + \beta_4}} b_{\beta_1}(0) b_{\beta_2}(0) b_{\beta_3}(0) b_{\beta_4}(0) \\
&= \sum_{\substack{\beta=(\beta_1, \beta_2, \beta_3, \beta_4) \in \mathbb{N}^4 \\ \beta_1 + \beta_2 = n = \beta_3 + \beta_4}} \liminf_{x \rightarrow 0} \{b_{\beta_1}(x) b_{\beta_2}(x) b_{\beta_3}(x) b_{\beta_4}(x)\} \\
&\leq \liminf_{x \rightarrow 0} \sum_{\substack{\beta=(\beta_1, \beta_2, \beta_3, \beta_4) \in \mathbb{N}^4 \\ \beta_1 + \beta_2 = n = \beta_3 + \beta_4}} b_{\beta_1}(x) b_{\beta_2}(x) b_{\beta_3}(x) b_{\beta_4}(x) \\
&= \liminf_{x \rightarrow 0} \sum_{n=2}^{\infty} \left| \sum_{k=1}^{n-1} b_{n-k}(x) b_k(x) \right|^2 = \liminf_{x \rightarrow 0} \|w(x, \cdot)\|_{L^4(\mathbb{T})}^4.
\end{aligned}$$

Thus we have $\lim_{x \rightarrow 0} \|w(x, \cdot)\|_{L^4(\mathbb{T})}^4 = \infty$, which implies that $\|w\|_{L^\infty(I \times \mathbb{T})} = \infty$.

Thus $u(x, y, t) \equiv \sum_{N=0}^{\infty} \frac{t^{2N}}{(2N)!} w_N(x, y)$ is a weak solution of $\mathcal{L}u = 0$ in $I \times \mathbb{T} \times (-\varepsilon, \varepsilon)$ with $\|u\|_{L^\infty(I \times \mathbb{T} \times (-\alpha, \alpha))} = \infty$ provided $0 < \varepsilon < \alpha < \alpha' \leq \frac{1}{4}$. ■

Part 5

Appendix

We include three results tangential to our development here. First we show that our hypoellipticity theorem for quasilinear equations doesn't generalize to more fully nonlinear equations, even with a degeneracy like that in (1.8) above. Then we show that almost generic Young functions have a remarkable recursive form that permits easy calculation of its iterates. Finally, we compute the Fedii operator \mathcal{L} in metric polar coordinates, and show that there are no nonconstant radial functions u for which $\mathcal{L}u$ is also radial.

A Monge-Ampère example

Let $\varphi(s)$ be a smooth even strictly convex function that vanishes only at $s = 0$, and vanishes to infinite order there. Then we have

$$|\varphi'(s)|^2 \leq \|\varphi''\|_\infty \varphi(s) \ll \varphi(s).$$

Now define $u(x, y) \equiv x^2 + \varphi(x)y^2$ and compute

$$\begin{aligned} D^2 u(x, y) &= \begin{bmatrix} u_{xx} & u_{xy} \\ u_{yx} & u_{yy} \end{bmatrix} = \begin{bmatrix} 2 + \varphi''(x)y^2 & 2\varphi'(x)y \\ 2\varphi'(x)y & 2\varphi(x) \end{bmatrix}; \\ \det D^2 u(x, y) &= 4\varphi(x) + 2\varphi(x)\varphi''(x)y^2 - 4\varphi'(x)^2 y^2. \end{aligned}$$

Then with

$$f(x, y) \equiv (2 + \varphi''(x)y^2)2\varphi(x) - 4\varphi'(x)^2 y^2$$

we have for (x, y) small enough that u is a convex solution to the Monge-Ampère equation

$$(11.1) \quad \det D^2 u(x, y) = f(x, y)$$

where f is smooth and positive away from $y = 0$ and satisfies

$$\frac{\partial}{\partial y} f(x, y) = o(f(x, y)).$$

Now we modify u by changing the multiple of x^2 on either side of the y -axis, which has little effect on $\det D^2 u(x, y)$:

$$\begin{aligned} u(x, y) &\equiv \begin{cases} x^2 + \varphi(x)y^2 & \text{if } x \geq 0 \\ \frac{1}{2}x^2 + \varphi(x)y^2 & \text{if } x \leq 0 \end{cases}, \\ \det D^2 u(x, y) &= \begin{cases} (2 + \varphi''(x)y^2)2\varphi(x) - 4\varphi'(x)^2 y^2 & \text{if } x \geq 0 \\ (1 + \varphi''(x)y^2)2\varphi(x) - \varphi'(x)^2 y^2 & \text{if } x \leq 0 \end{cases}. \end{aligned}$$

Thus with $f(x, y) \equiv \det D^2 u(x, y)$, we still have that f is smooth and positive away from $y = 0$ and satisfies

$$\frac{\partial}{\partial y} f(x, y) = o(f(x, y)).$$

But now $u \in C^{1,1} \setminus C^2$ is a nonsmooth solution to the Monge-Ampère equation $\det D^2 u = f$. Of course u is constant on the y axis, and the existence of this ‘Pogorelov segment’ accounts for the singularity of the solution - see [SaW].

The partial Legendre transform exhibits a close connection between this equation and quasilinear equations of the type considered in Theorem 1. Indeed, if u solves (11.1), then the associated

partial Legendre transform

$$\begin{aligned} s &= x \text{ and } t = u_y(x, y), \\ z &= u_x(x, y) \text{ and } v = y, \end{aligned}$$

satisfies the ‘Cauchy-Riemann’ equations,

$$\begin{aligned} z_s &= f v_t, \\ z_t &= -v_s, \end{aligned}$$

and hence v is a weak solution of the quasilinear equation

$$\left\{ \frac{\partial^2}{\partial s^2} + \frac{\partial}{\partial t} f(s, v(s, t)) \frac{\partial}{\partial t} \right\} v = 0,$$

of the form $\mathcal{L}_{\text{quasi}}$ in Theorem 1. The transform has nonnegative Jacobian

$$\frac{\partial(s, t)}{\partial(x, y)} = \det \begin{bmatrix} 1 & 0 \\ u_{yx} & u_{yy} \end{bmatrix} = u_{yy},$$

which is positive where f is positive. But $kv_t = z_s = u_{xx}$ has a discontinuity on the t -axis, and it follows easily that both $v_t = \frac{1}{f} z_s$ blows up at the t -axis, and that v has a discontinuity across the t -axis. Of course $v = y$ is bounded. The resolution here is that the partial Legendre transform is not one-to-one on the y -axis, and in fact the transformed equation is not valid at $x = 0$.

A criterion for a recursing formula with concave generator

Our Moser iteration above was rendered computable by using the special form Young function

$$\Phi_m(t) = e^{\left((\ln t)^{\frac{1}{m}} + 1\right)^m}.$$

The point is that this function has the *recursing form*

$$\Phi(t) = e^{g^{-1}(g(\ln t)+1)}, \quad g(s) = s^{\frac{1}{m}},$$

in which the iterates $\Phi^{(n)}(t)$ are given simply by

$$\Phi^{(n)}(t) = e^{g^{-1}(g(\ln t)+n)}.$$

Indeed,

$$\Phi \circ \Phi(t) = e^{g^{-1}(g(\ln \Phi(t))+1)} = e^{g^{-1}(g([g^{-1}(g(\ln t)+1)])+1)} = e^{g^{-1}(g(\ln t)+2)},$$

etc.

We turn here to the problem of deciding which strictly increasing functions $\Phi(t)$ can be expressed in the recursing form

$$\Phi(t) = e^{g^{-1}(g(\ln t)+1)}$$

for t large with a concave generator g . We have the following proposition.

PROPOSITION 116. *Suppose that $\Phi(t)$ is positive, increasing, convex and satisfies*

$$(12.1) \quad \frac{\Phi(t)}{t\Phi'(t)} \leq 1, \quad \text{for } t \text{ large.}$$

Then $\Phi(t) = e^{g^{-1}(g(\ln t)+1)}$ for t large with a concave generator g . In fact we may take a large and

$$g(s) = G_{\text{triv}}(e^s) \equiv G(a_0) + \frac{\ln \frac{t}{a_0}}{\ln \frac{a_1}{a_0}}, \quad a \leq e^s < \Phi(a),$$

and then extend g by the formula

$$g(\ln \Phi(t)) = g(\ln t) + 1, \quad t \geq a.$$

PROOF. With $G(t) = g(\ln t)$, we write

$$\begin{aligned} g(\ln \Phi(t)) &= g(\ln t) + 1; \\ G(\Phi(t)) &= G(t) + 1, \end{aligned}$$

and consider a starting point $a > 0$. Then we consider the orbit

$$\mathcal{O}(a) \equiv \{a_n\}_{n=1}^{\infty} = \left\{ \Phi^{(n)}(a) \right\}_{n=1}^{\infty}$$

of iterates of Φ starting at a , and define G on the orbit $\mathcal{O}(a)$ to satisfy the recursion

$$G(a_n) = G(\Phi(a_{n-1})) = G(a_{n-1}) + 1, \quad n \geq 1,$$

where the initial value $G(a_0) = G(a)$ is at our disposal. We obtain

$$G(a_n) = G(a) + n, \quad n \geq 0.$$

Consider a piecewise differentiable function Φ on an interval I with derivative $\Phi' > 1$ on I . Then Φ is strictly convex on I if and only if

$$x_2 - 2x_1 + x_0 \geq 0 \implies \Phi(x_2) - 2\Phi(x_1) + \Phi(x_0) > 0, \quad x_j \in I.$$

It now follows by induction on n that for Φ strictly convex with $\Phi' > 1$ we have

$$a_{n+1} - 2a_n + a_{n-1} = \Phi(a_n) - 2\Phi(a_{n-1}) + \Phi(a_{n-2}) > 0, \quad n \geq 2.$$

Since Φ is strictly increasing, we have $a < t < b \implies a_n < t_n < b_n$ for all $n \geq 0$. We now take $b = \Phi(a)$ and note that $\mathcal{O}(b) = \{b_n\}_{n=0}^\infty = \{a_{n+1}\}_{n=0}^\infty$ where $\mathcal{O}(a) = \{a_n\}_{n=0}^\infty$. Thus for any definition of G on $[a_0, a_1)$ we can uniquely extend G to $[a, \infty)$ by the formula

$$G(s) = G(t_n) = G(t) + n \text{ if } a_n \leq s < a_{n+1} \text{ and } s = t_n,$$

so as to satisfy the identity

$$G(\Phi(t)) = G(t) + 1, \quad t \geq a.$$

Then the function $g(s) = G(e^s)$ satisfies

$$g(\ln \Phi(t)) = g(\ln t) + 1, \quad t \geq a.$$

We now wish to choose an initial definition of $g(s) = G(e^s)$ on $[a_0, a_1)$ so that $g(s)$ on $[a_0, \infty)$ is concave and piecewise differentiable. Since $g'(s) = G'(e^s)e^s$ we see that g will be concave if and only if

$$(12.2) \quad tG'(t) \text{ is a decreasing function of } t.$$

Suppose a function G is defined on the initial segment $[a_0, a_1)$ and satisfies (12.2) on $[a_0, a_1)$ and $G(a_1) = G(a_0) + 1$. For example, if we require in addition that $tG'(t)$ is a constant C , then the choice $G(t) = G(a_0) + C \int_{a_0}^t \frac{1}{x} dx = G(a_0) + C \ln \frac{t}{a_0}$ trivially satisfies (12.2), and matches up at the orbit point a_1 provided

$$G(a_0) + C \ln \frac{a_1}{a_0} = G(a_1) = G(a_0) + 1; \quad C = \frac{1}{\ln \frac{a_1}{a_0}}.$$

We denote by $G_{\text{triv}}(t) = G(a_0) + \frac{\ln \frac{t}{a_0}}{\ln \frac{a_1}{a_0}}$ this trivial choice of G on $[a_0, a_1)$.

Now if G is any function on $[a_0, a_1)$ satisfying (12.2) and $G(a_1) = G(a_0) + 1$, then the extension of G to $[a_0, \infty)$ will satisfy (12.2) on each interval $[a_n, a_{n+1})$ provided that $\frac{\Phi(t)}{t\Phi'(t)}$ is a decreasing function of t . But this always holds if Φ is positive increasing and convex. Indeed, on the next interval $[a_1, a_2)$ we have

$$G(s) = G(\Phi(t)) = G(t) + 1 \text{ if } a_1 \leq s = \Phi(t) < a_2.$$

Thus $G'(s) = \frac{d}{ds} (G(\Phi^{-1}(s)) + 1) = \frac{G'(\Phi^{-1}(s))}{\Phi'(\Phi^{-1}(s))}$ and

$$sG'(s) = s \frac{G'(\Phi^{-1}(s))}{\Phi'(\Phi^{-1}(s))} = \frac{G'(t)}{\Phi'(t)} = \frac{\Phi(t)}{t\Phi'(t)} tG'(t)$$

will be decreasing provided both $\frac{\Phi(t)}{t\Phi'(t)}\Phi$ and $tG'(t)$ are. Now an induction on n shows that G satisfies (12.2) on each interval $[a_n, a_{n+1})$.

It remains only to check that (12.2) holds at the orbit points a_n for a suitable choice of G . But we claim that (12.2) holds for the trivial choice $G_{\text{triv}}(t) = G(a_0) + \frac{\ln \frac{t}{a_0}}{\ln \frac{a_1}{a_0}}$. Indeed, at the orbit point a_1 we have

$$\begin{aligned} \lim_{\substack{t \rightarrow a_1 \\ t < a_1}} tG'_{\text{triv}}(t) - \lim_{\substack{t \rightarrow a_1 \\ t > a_1}} tG'_{\text{triv}}(t) &= a_1 \frac{\frac{1}{a_1}}{\ln \frac{a_1}{a_0}} - a_1 \lim_{\substack{t \rightarrow a_1 \\ t > a_1}} \frac{d}{dt} G_{\text{triv}}(\Phi^{-1}(t)) \\ &= \frac{1}{\ln \frac{a_1}{a_0}} - a_1 \lim_{\substack{t \rightarrow a_1 \\ t > a_1}} \frac{G'_{\text{triv}}(\Phi^{-1}(t))}{\Phi'(\Phi^{-1}(t))} \\ &= \frac{1}{\ln \frac{a_1}{a_0}} - a_1 \frac{G'_{\text{triv}}(a_0)}{\Phi'(a_0)} \\ &= \frac{1}{\ln \frac{a_1}{a_0}} - a_1 \frac{1}{\Phi'(a_0)} \frac{\frac{1}{a_0}}{\ln \frac{a_1}{a_0}} \\ &= \frac{1}{\ln \frac{a_1}{a_0}} \left\{ 1 - \frac{\frac{a_1}{a_0}}{\Phi'(a_0)} \right\} \geq 0 \end{aligned}$$

provided $\frac{\frac{a_1}{a_0}}{\Phi'(a_0)} \leq 1$. Similarly, at the orbit point a_n we have

$$\lim_{\substack{t \rightarrow a_n \\ t < a_n}} tG'_{\text{triv}}(t) - \lim_{\substack{t \rightarrow a_n \\ t > a_n}} tG'_{\text{triv}}(t) \geq 0$$

provided

$$\frac{\frac{a_n}{a_{n-1}}}{\Phi'(a_{n-1})} \leq 1; \quad \frac{\Phi(a_{n-1})}{a_{n-1}\Phi'(a_{n-1})} \leq 1,$$

which is implied by (12.1). ■

Absence of radial solutions

Recall our family of geometries with inverse metric tensor $A = \begin{bmatrix} 1 & 0 \\ 0 & f(x)^2 \end{bmatrix}$ and A -distance dt given by $dt^2 = dx^2 + \frac{1}{f(x)^2} dy^2$, which coincides with the familiar control metric d associated with A . Here we suppose that $f(x)$ is positive away from zero and $f(0) = 0$ (thus prohibiting the usual elliptic geometry). We say that a function $v(x, y)$ is radial if it depends only on the metric distance $r = d((0, 0), (x, y))$ from the point (x, y) to the origin. Here we show that there is **no** nonconstant radial solution to the equation $\mathcal{L}v = \varphi(r)$ with radial right hand side for such geometries. Note that this includes all of the finite type geometries of this form, as well as the infinitely degenerate ones.

We work in Region 1 for convenience. Recall that

$$\begin{aligned} \frac{\partial(x, y)}{\partial(r, \lambda)} &= \begin{bmatrix} \frac{\sqrt{\lambda^2 - f(x)^2}}{\lambda} & \frac{\sqrt{\lambda^2 - f(x)^2}}{\lambda} \cdot \int_0^x \frac{f(u)^2}{(\lambda^2 - f(u)^2)^{\frac{3}{2}}} du \\ \frac{f(x)^2}{\lambda} & \frac{f(x)^2 - \lambda^2}{\lambda} \cdot \int_0^x \frac{f(u)^2}{(\lambda^2 - f(u)^2)^{\frac{3}{2}}} du \end{bmatrix} \\ &= \begin{bmatrix} \frac{\sqrt{\lambda^2 - f(x)^2}}{\lambda} & \frac{\sqrt{\lambda^2 - f(x)^2}}{\lambda} m_3(x) \\ \frac{f(x)^2}{\lambda} & \frac{f(x)^2 - \lambda^2}{\lambda} m_3(x) \end{bmatrix}, \end{aligned}$$

where we write

$$(13.1) \quad m_k(x) = \int_0^x \frac{f(u)^2}{(\lambda^2 - f(u)^2)^{\frac{k}{2}}} du.$$

Then $\det \left(\frac{\partial(x, y)}{\partial(r, \lambda)} \right) = -\sqrt{\lambda^2 - f(x)^2} m_3(x)$, and the inverse matrix is given by

$$\begin{aligned} \frac{\partial(r, \lambda)}{\partial(x, y)} &= \frac{1}{\det \frac{\partial(x, y)}{\partial(r, \lambda)}} \begin{bmatrix} \frac{\sqrt{\lambda^2 - f(x)^2}}{\lambda} & \frac{\sqrt{\lambda^2 - f(x)^2}}{\lambda} m_3(x) \\ \frac{f(x)^2}{\lambda} & \frac{f(x)^2 - \lambda^2}{\lambda} m_3(x) \end{bmatrix} \\ &= \begin{bmatrix} \frac{\sqrt{\lambda^2 - f(x)^2}}{\lambda} & \frac{1}{\lambda} \\ \frac{f(x)^2}{\lambda \sqrt{\lambda^2 - f(x)^2} m_3(x)} & -\frac{1}{\lambda m_3(x)} \end{bmatrix}. \end{aligned}$$

Thus

$$\begin{aligned}\frac{\partial r}{\partial x} &= \frac{\sqrt{\lambda^2 - f(x)^2}}{\lambda} & \frac{\partial r}{\partial y} &= \frac{1}{\lambda} \\ \frac{\partial \lambda}{\partial x} &= \frac{f(x)^2}{\sqrt{\lambda^2 - f(x)^2} \lambda m_3(x)} & \frac{\partial \lambda}{\partial y} &= -\frac{1}{\lambda m_3(x)},\end{aligned}$$

and so

$$\begin{aligned}\frac{\partial}{\partial x} &= \frac{\partial \lambda}{\partial x} \frac{\partial}{\partial \lambda} + \frac{\partial r}{\partial x} \frac{\partial}{\partial r} \\ &= \frac{f(x)^2}{\sqrt{\lambda^2 - f(x)^2} \lambda m_3(x)} \frac{\partial}{\partial \lambda} + \frac{\sqrt{\lambda^2 - f(x)^2}}{\lambda} \frac{\partial}{\partial r}, \quad \text{and} \\ \frac{\partial}{\partial y} &= \frac{\partial \lambda}{\partial y} \frac{\partial}{\partial \lambda} + \frac{\partial r}{\partial y} \frac{\partial}{\partial r} = -\frac{1}{\lambda \int_0^x \frac{f(u)^2}{(\lambda^2 - f(u)^2)^{\frac{3}{2}}} du} \frac{\partial}{\partial \lambda} + \frac{1}{\lambda} \frac{\partial}{\partial r}.\end{aligned}$$

REMARK 117. If $v = v(r)$ is radial, then

$$\begin{aligned}\nabla_A v &= \left(\frac{\partial v}{\partial x}, f(x) \frac{\partial v}{\partial y} \right) = \left(\frac{\sqrt{\lambda^2 - f(x)^2}}{\lambda} \frac{\partial v}{\partial r}, f(x) \frac{1}{\lambda} \frac{\partial v}{\partial r} \right) \\ &= \left(\sqrt{1 - \left(\frac{f(x)}{\lambda} \right)^2}, \frac{f(x)}{\lambda} \right) \frac{\partial v}{\partial r},\end{aligned}$$

which is not in general radial, but its modulus is:

$$|\nabla_A v| = \left| \frac{\partial v}{\partial r} \right|.$$

Note also that $\frac{f(x)}{\lambda} = \frac{f(x)}{f(X(\lambda))}$ where $(X(\lambda), Y(\lambda))$ is the turning point of the geodesic through the origin and (x, y) .

Now we compute the second derivatives:

$$\begin{aligned}\frac{\partial^2 r}{\partial x^2} &= \frac{\partial}{\partial x} \frac{\sqrt{\lambda^2 - f(x)^2}}{\lambda} \\ &= -\frac{\sqrt{\lambda^2 - f(x)^2}}{\lambda^2} \frac{\partial \lambda}{\partial x} + \frac{\lambda \frac{\partial \lambda}{\partial x} - f(x) f'(x)}{\lambda \sqrt{\lambda^2 - f(x)^2}} \\ &= -\frac{f(x)^2}{\lambda^3 m_3(x)} + \frac{f(x)^2}{\lambda (\lambda^2 - f(x)^2) m_3(x)} - \frac{f(x) f'(x)}{\lambda \sqrt{\lambda^2 - f(x)^2}} \\ &\quad \text{and} \\ \frac{\partial^2 r}{\partial y^2} &= \frac{\partial}{\partial y} \frac{1}{\lambda} = -\frac{1}{\lambda^2} \frac{\partial \lambda}{\partial y} = \frac{1}{\lambda^3 m_3(x)}.\end{aligned}$$

We then have

$$\begin{aligned}
\frac{\partial^2 v}{\partial x^2} &= \frac{\partial}{\partial x} \left(\frac{\partial \lambda}{\partial x} v_\lambda(r, \lambda) + \frac{\partial r}{\partial x} v_r(r, \lambda) \right) \\
&= \frac{\partial \lambda}{\partial x} \frac{\partial}{\partial x} v_\lambda(r, \lambda) + \frac{\partial r}{\partial x} \frac{\partial}{\partial x} v_r(r, \lambda) \\
&\quad + \frac{\partial^2 \lambda}{\partial x^2} v_\lambda(r, \lambda) + \frac{\partial^2 r}{\partial x^2} v_r(r, \lambda) \\
&= \frac{\partial \lambda}{\partial x} \left(\frac{\partial \lambda}{\partial x} \frac{\partial}{\partial \lambda} + \frac{\partial r}{\partial x} \frac{\partial}{\partial r} \right) v_\lambda(r, \lambda) + \frac{\partial r}{\partial x} \left(\frac{\partial \lambda}{\partial x} \frac{\partial}{\partial \lambda} + \frac{\partial r}{\partial x} \frac{\partial}{\partial r} \right) v_r(r, \lambda) \\
&\quad + \frac{\partial^2 \lambda}{\partial x^2} v_\lambda(r, \lambda) + \frac{\partial^2 r}{\partial x^2} v_r(r, \lambda) \\
&= \left(\frac{\partial \lambda}{\partial x} \right)^2 v_{\lambda\lambda}(r, \lambda) + 2 \frac{\partial r}{\partial x} \frac{\partial \lambda}{\partial x} v_{r\lambda}(r, \lambda) + \left(\frac{\partial r}{\partial x} \right)^2 v_{rr}(r, \lambda) \\
&\quad + \frac{\partial^2 \lambda}{\partial x^2} v_\lambda(r, \lambda) + \frac{\partial^2 r}{\partial x^2} v_r(r, \lambda),
\end{aligned}$$

and similarly,

$$\begin{aligned}
\frac{\partial^2 v}{\partial y^2} &= \left(\frac{\partial \lambda}{\partial y} \right)^2 v_{\lambda\lambda}(r, \lambda) + 2 \frac{\partial r}{\partial y} \frac{\partial \lambda}{\partial y} v_{r\lambda}(r, \lambda) + \left(\frac{\partial r}{\partial y} \right)^2 v_{rr}(r, \lambda) \\
&\quad + \frac{\partial^2 \lambda}{\partial y^2} v_\lambda(r, \lambda) + \frac{\partial^2 r}{\partial y^2} v_r(r, \lambda).
\end{aligned}$$

Then, if $v = v(r, \lambda)$, we have the following expression for \mathcal{L} in polar coordinates:

$$\begin{aligned}
\mathcal{L}v &= \frac{\partial^2 v}{\partial x^2} + f(x)^2 \frac{\partial^2 v}{\partial y^2} \\
&= \left(\left(\frac{\partial \lambda}{\partial x} \right)^2 + f(x)^2 \left(\frac{\partial \lambda}{\partial y} \right)^2 \right) v_{\lambda\lambda}(r, \lambda) + \left(\left(\frac{\partial r}{\partial x} \right)^2 + f(x)^2 \left(\frac{\partial r}{\partial y} \right)^2 \right) v_{rr}(r, \lambda) \\
&\quad + 2 \left(\frac{\partial r}{\partial x} \frac{\partial \lambda}{\partial x} + f(x)^2 \frac{\partial r}{\partial y} \frac{\partial \lambda}{\partial y} \right) v_{r\lambda}(r, \lambda) \\
&\quad + (\mathcal{L}\lambda) v_\lambda(r, \lambda) + (\mathcal{L}r) v_r(r, \lambda).
\end{aligned}$$

Since

$$\begin{aligned}
\left(\frac{\partial \lambda}{\partial x}\right)^2 + f(x)^2 \left(\frac{\partial \lambda}{\partial y}\right)^2 &= \left(\frac{f(x)^2}{\sqrt{\lambda^2 - f(x)^2} \lambda m_3(x)}\right)^2 + f(x)^2 \left(-\frac{1}{\lambda m_3(x)}\right)^2 \\
&= \frac{f(x)^4}{(\lambda^2 - f(x)^2) \lambda^2 m_3(x)^2} + \frac{f(x)^2}{\lambda^2 m_3(x)^2} \\
&= \frac{f(x)^4 + (\lambda^2 - f(x)^2) f(x)^2}{(\lambda^2 - f(x)^2) \lambda^2 m_3(x)^2} \\
&= \frac{f(x)^2}{(\lambda^2 - f(x)^2) m_3(x)^2}, \\
\left(\frac{\partial r}{\partial x}\right)^2 + f(x)^2 \left(\frac{\partial r}{\partial y}\right)^2 &= \left(\frac{\sqrt{\lambda^2 - f(x)^2}}{\lambda}\right)^2 + f(x)^2 \left(\frac{1}{\lambda}\right)^2 \\
&= \frac{\lambda^2 - f(x)^2}{\lambda^2} + \frac{f(x)^2}{\lambda^2} = 1, \\
\frac{\partial r}{\partial x} \frac{\partial \lambda}{\partial x} + f(x)^2 \frac{\partial r}{\partial y} \frac{\partial \lambda}{\partial y} &= \frac{\sqrt{\lambda^2 - f(x)^2}}{\lambda} \frac{f(x)^2}{\sqrt{\lambda^2 - f(x)^2} \lambda m_3(x)} - f(x)^2 \frac{1}{\lambda} \frac{1}{\lambda m_3(x)} \\
&= \frac{f(x)^2}{\lambda^2 m_3(x)} - \frac{f(x)^2}{\lambda^2 m_3(x)} = 0,
\end{aligned}$$

we obtain

$$\begin{aligned}
\mathcal{L}v &= \frac{\partial^2 v}{\partial x^2} + f(x)^2 \frac{\partial^2 v}{\partial y^2} \\
&= \frac{f(x)^2}{(\lambda^2 - f(x)^2) m_3(x)^2} v_{\lambda\lambda}(r, \lambda) + v_{rr}(r, \lambda) \\
&\quad + (\mathcal{L}\lambda) v_\lambda(r, \lambda) + (\mathcal{L}r) v_r(r, \lambda).
\end{aligned}$$

In particular, if $v = v(r)$ is radial,

$$(13.2) \quad \mathcal{L}v(r) = v_{rr}(r) + (\mathcal{L}r) v_r(r).$$

In order to understand equation (13.2), we must compute $\mathcal{L}r$,

$$\begin{aligned}
 (13.3) \quad \mathcal{L}r &= \frac{\partial^2 r}{\partial x^2} + f(x)^2 \frac{\partial^2 r}{\partial y^2} = \left(\frac{\partial}{\partial x} \frac{\sqrt{\lambda^2 - f(x)^2}}{\lambda} \right) + \left(\frac{\partial}{\partial y} \frac{f(x)^2}{\lambda} \right) \\
 &= \frac{\lambda \frac{\partial \lambda}{\partial x} - f(x) f'(x)}{\lambda \sqrt{\lambda^2 - f(x)^2}} - \frac{\partial \lambda}{\partial x} \frac{\sqrt{\lambda^2 - f(x)^2}}{\lambda^2} - \frac{\partial \lambda}{\partial y} \frac{f(x)^2}{\lambda^2} \\
 &= \frac{\lambda \frac{f(x)^2}{\sqrt{\lambda^2 - f(x)^2} \lambda m_3(x)} - f(x) f'(x)}{\lambda \sqrt{\lambda^2 - f(x)^2}} - \frac{1}{\lambda} \left(\frac{\partial \lambda}{\partial x} \frac{\partial r}{\partial x} + f(x)^2 \frac{\partial \lambda}{\partial y} \frac{\partial r}{\partial y} \right) \\
 &= \frac{f(x)^2 - f(x) f'(x) m_3(x) \sqrt{\lambda^2 - f(x)^2}}{\lambda (\lambda^2 - f(x)^2) m_3(x)} \\
 &= \frac{f(x)^2}{\lambda (\lambda^2 - f(x)^2) m_3(x)} - \frac{f(x) f'(x)}{\lambda \sqrt{\lambda^2 - f(x)^2}},
 \end{aligned}$$

where we used that $\frac{\partial \lambda}{\partial x} \frac{\partial r}{\partial x} + f(x)^2 \frac{\partial \lambda}{\partial y} \frac{\partial r}{\partial y} = 0$. Now we note that since

$$\frac{\partial}{\partial r} = \frac{\sqrt{\lambda^2 - f(x)^2}}{\lambda} \frac{\partial}{\partial x} + \frac{f(x)^2}{\lambda} \frac{\partial}{\partial y},$$

we have

$$\begin{aligned}
 \frac{\partial}{\partial r} \left(\sqrt{\lambda^2 - f(x)^2} \right) &= \frac{\sqrt{\lambda^2 - f(x)^2}}{\lambda} \frac{\partial}{\partial x} \left(\sqrt{\lambda^2 - f(x)^2} \right) + \frac{f(x)^2}{\lambda} \frac{\partial}{\partial y} \left(\sqrt{\lambda^2 - f(x)^2} \right) \\
 &= \frac{\sqrt{\lambda^2 - f(x)^2}}{\lambda} \frac{\lambda \frac{\partial \lambda}{\partial x} - f(x) f'(x)}{\sqrt{\lambda^2 - f(x)^2}} + \frac{f(x)^2}{\lambda} \frac{\lambda \frac{\partial \lambda}{\partial y}}{\sqrt{\lambda^2 - f(x)^2}} \\
 &= \frac{1}{\lambda} \left(\lambda \frac{f(x)^2}{\sqrt{\lambda^2 - f(x)^2} \lambda m_3(x)} - f(x) f'(x) \right) - \frac{f(x)^2}{\lambda} \frac{\lambda \frac{1}{\lambda m_3(x)}}{\sqrt{\lambda^2 - f(x)^2}} \\
 &= \frac{f(x)^2}{\sqrt{\lambda^2 - f(x)^2} \lambda m_3(x)} - \frac{f(x) f'(x)}{\lambda} - \frac{f(x)^2}{\lambda \sqrt{\lambda^2 - f(x)^2} m_3(x)} = -\frac{f(x) f'(x)}{\lambda},
 \end{aligned}$$

so that

$$(13.4) \quad \frac{\partial}{\partial r} \left(\ln \sqrt{\lambda^2 - f(x)^2} \right) = -\frac{f(x) f'(x)}{\lambda \sqrt{\lambda^2 - f(x)^2}}.$$

We also have

$$\begin{aligned}
 \frac{\partial m_3(x)}{\partial x} &= \frac{\partial}{\partial x} \int_0^x \frac{f(u)^2}{(\lambda^2 - f(u)^2)^{\frac{3}{2}}} du \\
 &= \frac{f(x)^2}{(\lambda^2 - f(x)^2)^{\frac{3}{2}}} - 3\lambda \frac{\partial \lambda}{\partial x} \int_0^x \frac{f(u)^2}{(\lambda^2 - f(u)^2)^{\frac{5}{2}}} du \\
 (13.5) \quad &= \frac{f(x)^2}{(\lambda^2 - f(x)^2)^{\frac{3}{2}}} - 3 \frac{f(x)^2 m_5(x)}{\sqrt{\lambda^2 - f(x)^2} m_3(x)},
 \end{aligned}$$

$$\begin{aligned}
 \frac{\partial m_3(x)}{\partial y} &= \frac{\partial}{\partial y} \int_0^x \frac{f(u)^2}{(\lambda^2 - f(u)^2)^{\frac{3}{2}}} du \\
 &= -3\lambda \frac{\partial \lambda}{\partial y} \int_0^x \frac{f(u)^2}{(\lambda^2 - f(u)^2)^{\frac{5}{2}}} du \\
 (13.6) \quad &= 3 \frac{m_5(x)}{m_3(x)},
 \end{aligned}$$

and hence

$$\begin{aligned}
 \frac{\partial}{\partial r} m_3(x) &= \frac{\sqrt{\lambda^2 - f(x)^2}}{\lambda} \frac{\partial m_3(x)}{\partial x} + \frac{f(x)^2}{\lambda} \frac{\partial m_3(x)}{\partial y} \\
 &= \frac{\sqrt{\lambda^2 - f(x)^2}}{\lambda} \frac{f(x)^2}{(\lambda^2 - f(x)^2)^{\frac{3}{2}}} - 3 \frac{\sqrt{\lambda^2 - f(x)^2}}{\lambda} m_5(x) \lambda \frac{\partial \lambda}{\partial x} - 3 \frac{f(x)^2}{\lambda} m_5(x) \lambda \frac{\partial \lambda}{\partial y} \\
 &= \frac{\sqrt{\lambda^2 - f(x)^2}}{\lambda} \frac{f(x)^2}{(\lambda^2 - f(x)^2)^{\frac{3}{2}}} - 3 \frac{\sqrt{\lambda^2 - f(x)^2}}{\lambda} m_5(x) \lambda \frac{f(x)^2}{\sqrt{\lambda^2 - f(x)^2} \lambda m_3(x)} \\
 &\quad + 3 \frac{f(x)^2}{\lambda} m_5(x) \lambda \frac{1}{\lambda m_3(x)} \\
 &= \frac{f(x)^2}{\lambda (\lambda^2 - f(x)^2)} - 3 \frac{f(x)^2 m_5(x)}{\lambda m_3(x)} + 3 \frac{f(x)^2 m_5(x)}{\lambda m_3(x)} \\
 &= \frac{f(x)^2}{\lambda (\lambda^2 - f(x)^2)},
 \end{aligned}$$

so that

$$(13.7) \quad \frac{\partial}{\partial r} \ln(m_3(x)) = \frac{f(x)^2}{\lambda (\lambda^2 - f(x)^2) m_3(x)}.$$

From (13.3), (13.4), and (13.7), we finally obtain

$$\begin{aligned}
 (13.8) \quad \mathcal{L}r &= \frac{\partial}{\partial r} \ln(m_3(x)) + \frac{\partial}{\partial r} \left(\ln \sqrt{\lambda^2 - f(x)^2} \right) \\
 &= \frac{\partial}{\partial r} \ln \left(\sqrt{\lambda^2 - f(x)^2} m_3(x) \right).
 \end{aligned}$$

Substituting this into (13.2), we see that if $v(r)$ is a radial solution of $\mathcal{L}v = \varphi(r, \lambda)$, then

$$(13.9) \quad \frac{1}{\sqrt{\lambda^2 - f(x)^2} m_3(x)} \frac{\partial}{\partial r} \left(\sqrt{\lambda^2 - f(x)^2} m_3(x) \frac{\partial v}{\partial r} \right) = \varphi(r, \lambda).$$

CONCLUSION 118. *If $v = v(r)$ is a radial solution of $\mathcal{L}v = 0$, then from (13.9) it follows that for some constant C ,*

$$\frac{\partial v}{\partial r} = \frac{C}{\sqrt{\lambda^2 - f(x)^2} m_3(x)},$$

*and if $C \neq 0$, the function on the right is **not** radial. This means that there exist **no nonconstant radial solutions** to $\mathcal{L}v = 0$. Indeed, there is no nonconstant radial solution to $\mathcal{L}v = \varphi(r)$, since if v is a nonconstant radial solution, then from $\mathcal{L}v(r) = v''(r) + (\mathcal{L}r) v'(r)$ we see that $\mathcal{L}r$ is radial, and hence from (13.8) that $\sqrt{\lambda^2 - f(x)^2} m_3(x)$ is radial, a contradiction.*

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